KONINKLIJKE VLAAMSE ACADEMIE VAN BELGIE
VOOR WETENSCHAPPEN EN KUNSTEN

ACTUARIAL AND FINANCIAL MATHEMATICS CONFERENCE
Interplay between Finance and Insurance

February 7-8, 2008

Michèle Vanmaele, Griselda Deelstra, Ann De Schepper, Jan Dhaene, Huguette Reynaerts, Wim Schoutens & Paul Van Goethem (Eds.)

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Actuarial and Financial Mathematics Conference
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KONINKLIJKE VLAAMSE ACADEMIE VAN BELGIE
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PREFACE

This Contactforum "Actuarial and Financial Mathematics Conference (AFMathConf2008)" attracted many participants, both researchers and practitioners. This year, we counted 98 participants from 15 different countries.
Many young researchers, in particular Ph.D. students and Postdocs, working in the field of Financial and Actuarial Mathematics found their way to this event where they could discuss recent developments in the theory of mathematical finance and insurance and their interplay, and could interact with the guest speakers and practitioners.

We thank all our speakers and in particular the guest speakers: *Laura Ballotta (Cass Business School, City University UK)*, *Pauline Barrieu (London School of Economics, UK)*, *Nicole Bäuerle (Universität Karlsruhe, Germany)*, *Freddy Delbaen (ETH Zürich, Switzerland)*, *Paul Embrechts (ETH Zürich, Switzerland)*, *Farshid Jamshidian (University of Twente, The Netherlands)*, *Thomas Møller (PFA Pension, Denmark)* and *Antoon Pelsser (University of Amsterdam, The Netherlands),* for their enthusiasm and their interesting contributions which made this day a great success. Special thanks also go to the discussants: *Ann De Schepper*, *Michael Kunisch*, *Zbigniew Matosek*, *Ragnar Norberg*, *Antoon Pelsser*, *Tim Pillards*, *Grigory Temnov*, *Michel Verschuere*, for their valuable suggestions and critical comments on the presented contributed papers.
Eight out of nine papers related to the contributed talks are published in these proceedings. Also a summary of the talk of the guest speaker Professor Jamshidian (member of the Advisory Board of Advanced Mathematical Methods in Finance (AMaMeF)) can be found here.

We owe much gratitude to the members of the scientific committee, namely *Michel Denuit (Université Catholique de Louvain, Belgium)*, *Marc Goovaerts (Katholieke Universiteit Leuven, Belgium)*, *Rob Kaas (University of Amsterdam, The Netherlands)*, *Ragnar Norberg (London School of Economics, UK)*, *Bernt Øksendal (University of Oslo, Norway)* and *Noel Veraverbeke (Universiteit Hasselt, Belgium)* for the scientific support and to the administrative staff consisting of *Wouter Dewolf* and *Stefanie Vermeire*, both of the Department of Applied Mathematics and Computer Science of the Ghent University, for the administrative work.

Finally, we could not have hold this very rewarding event, academically as well as socially, without the financial support of our sponsors : Royal Flemish Academy of Belgium for Science and Arts, Research Foundation - Flanders (FWO), Scientific Research Network FWO “Fundamental Methods and Techniques in Mathematics”, Fonds de la Recherche Scientific (FNRS), KULeuven Fortis Chair in financial and actuarial risk management, Fortis Insurance Belgium, IAP P6/07 from the IAP programme (Belgian Scientific Policy) and ESF program Advanced Mathematical Methods in Finance (AMaMeF). We are greatly indebted to them.

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INVITED TALK

NUMERAIRE INVARIANCE AND APPLICATION TO OPTION PRICING AND HEDGING

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1. INTRODUCTION

Numeraire invariance is a well-known technique in option pricing and hedging theory. It takes a convenient asset as the *numeraire*, as if it were the medium of exchange, and expresses all other asset and option prices in units of this numeraire. Since the price of the numeraire relative to itself is identically 1 at all times, this reduces pricing and hedging to a market with zero-interest rates. A somewhat controversial implication is that the modelling focus should be more on the asset price *ratios* rather than on the asset price processes themselves.

The idea of numeraire invariance is already implicit in Merton (1973), and since then many authors have contributed to its development. After a brief survey of its origins, we state and prove the numeraire invariance principle for general semimartingale price processes, following essentially Duffie (2001). We then present its application to unique pricing in arbitrage-free models and discuss nondegeneracy and unique hedging.

Next, using numeraire invariance, we show that if the underlying asset *ratios* follow a diffusion, then a payoff that is a *homogeneous function* of the asset payoffs can *always* be replicated (subject to mild growth conditions) and hence also uniquely priced. The deltas (hedge ratios) are given by the partial derivatives of either the "projective option price function", or equivalently, of the "homogeneous option price function", either of which is the solution of a PDE. We illustrate the classical multivariate lognormal case from this angel.

To illustrate replication under the presence of jumps, we work out a little-known *exponential Poisson model*, first for the exchange option, and then for a multivariate generalization with an arbitrary homogeneous payoff function. Here, the option price function satisfies a partial *difference* equation, and the deltas are given by partial *differences*. We mention a connection to martingale representation, from which the explicit formulae are actually drawn.

[1]This paper is a short version of the paper Jamshidian (2007) on exchange options. For possible future updates visit wwwhome.math.utwente.nl/~jamshidianf

In the final section, we first highlight the role played by homogeneity, emphasizing that if the covariation matrix of the underlying assets is nondegenerate, then nonhomogeneous payoffs *cannot* be replicated. We then extend the discussion to assets with dividends. Finally, we derive the ubiquitous bivariate lognormal exchange option formula by a change of measure.

We will confine the discussion to European options with expiration denoted T.

2. A BRIEF SURVEY

2.1. Merton's extension of Black-Scholes

Let be given a zero-dividend asset with price process $A = (A_t)$. Let $C = (C_t)$ denote the price process of a call option on A with strike price K and expiration T, which we wish to find. So, the option payoff is

$$C_T = (A_T - K)^+.$$

To ***replicate*** C, another asset is needed. Black and Scholes (1973) take as the second asset a money market of the form e^{rt}. Merton's idea is to take the T-maturity zero-coupon bond B with principal K, i.e., $B_T = K$. The payoff can now be expressed in terms of both assets:

$$C_T = (A_T - B_T)^+.$$

The payoff's ***homogeneity*** allows one to factor out B:

$$F_T = (X_T - 1)^+,$$

where

$$X := \frac{A}{B}, \quad F := \frac{C}{B},$$

are the ***forward prices*** of the asset and the option. Merton (1973) argues that it is sufficient to replicate the forward option by trading the forward asset, i.e., to find a δ such that

$$dF_t = \delta_t dX_t.$$

The *same* δ should then serve as the hedge ratio with respect to asset A.

Assuming $F_t = f(t, X_t)$ for some f, by ***Itô's formula*** the equation $dF = \delta dX$ is equivalent to the following formula for δ_t and **PDE** for $f(t,x)$ with terminal condition $f(T,x) = (x-1)^+$:

$$\delta_t = \frac{\partial f}{\partial x}(t, X_t),$$

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma_t^2 x^2 \frac{\partial^2 f}{\partial x^2} = 0,$$

where σ_t is the ***forward-price volatility*** (assumed deterministic by Merton):

$$d[X]_t = \sigma_t^2 X_t^2 dt.$$

(The first (second) equation follows by equating the martingale (drift) terms of the two equations for dF.) Thus by "factoring out" asset B, the problem with a stochastic interest rate reduces to a call option struck at 1 in the Black-Scholes model with *zero interest rate*.

More generally, when asset A pays dividends at a constant rate y, the above applies with the forward asset price $X_t = e^{-y(T-t)} A_t / B_t$.

2.2. Margrabe's extension to exchange options

Margrabe (1978) showed that Merton's argument extends to an option to exchange any two assets A and B. His idea was to replicate the exchange option price process C according to the SDE

$$dC_t = \delta_t^A dA_t + \delta_t^B dB_t.$$

Assuming $C_t = c(t, A_t, B_t)$ for some function $c(t, a, b)$, he noted that by *Itô's formula* this equation is implied by the system of equations

$$\delta_t^A = \frac{\partial c}{\partial a}(t, A_t, B_t), \qquad \delta_t^B = \frac{\partial c}{\partial b}(t, A_t, B_t),$$

$$\frac{\partial c}{\partial t} + \frac{1}{2}\sigma_A^2 a^2 \frac{\partial^2 c}{\partial a^2} + \frac{1}{2}\sigma_B^2 b^2 \frac{\partial^2 c}{\partial b^2} + \sigma_A \sigma_B \rho\, ab \frac{\partial^2 c}{\partial a \partial b} = 0,$$

where σ_A and σ_B are the volatilities of A and B and ρ is their correlation, assumed constants. The converse is also true if $|\rho| \neq 1$. (Note however, this nondegeneracy condition excludes the Black-Scholes and 1-factor short-rate diffusion models).

Margrabe stated that if the option payoff is ***homogeneous*** of degree 1 in (a, b) (such as $(a-b)^+$ as in the case of an exchange option), then the PDE above should have a homogeneous solution $c(t, a, b)$. But then, *Euler's formula* for homogeneous functions implies $c = a\partial c/\partial a + b\partial c/\partial b$. Thus if we choose $\delta^A = \partial c/\partial a$ and $\delta^B = \partial c/\partial b$ as above, we get

$$C_t = \delta_t^A A_t + \delta_t^B B_t.$$

Together with the equation $dC = \delta^A dA + \delta^B dB$, this means these deltas are ***self financing***.

Merton had made similar observations and provided the homogeneous solution $c(t, a, b)$ of the above PDE by reducing it to the 1-dimensional PDE of Section 2.1 via the transformation

$$f(t, x) = c(t, a, b)/b = c(t, x, 1), \quad x = a/b,$$

with volatility σ in the 1-dimensional PDE given by that of ***asset ratio*** A/B:

$$\sigma^2 = \sigma_A^2 + \sigma_B^2 - 2\sigma_A \sigma_B \rho.$$

Coining the term ***numeraire***, Margrabe presented (acknowledging Stephen Ross) a financial interpretation of Merton's algebraic reduction. He proposed to measure the asset and option prices in terms of asset B, as in a barter economy where B serves as the medium of exchange. This provided the intuition behind Merton's reduction to zero interest rates.

Note, the exchange option is replicated here by dynamic trading in only assets A and B.

2.3. Equivalent martingale measures

Harrison and Kreps (1979) and Harrison and Pliska (1981) pioneered the application of martingale theory to option pricing. They showed that ***no-arbitrage*** in the sense of no *free lunches* is essentially equivalent to the existence of an equivalent measure under which discounted prices are martingales. (See Delbaen and Schachermayer (2006) for the general theory.) Options can thus be priced by computing the ***discounted payoff expectation***.

For discounting, they utilized the finite variation money market numeraire $\exp(\int_0^t r_s ds)$, where r_t is the instantaneous interest rate. This included the Black-Scholes and short-rate models, but did not address Merton's and Margrabe's approach where the numeraire had infinite variation. With the advent of the ***forward measure***, it was clear that the discounting could also be done with a zero-coupon bond, and this often simplified the calculation as discounting was in effect performed outside the expectation (e.g., Jamshidian (1993) and El-Karoui et al. (1995)).

Another useful numeraire, "the annuity", was used by Neuberger (1990) to price interest-rate swaptions. It serves as the industry standard to this date for quoting swaption volatilities. Eventually, El-Karoui et al. (1995) showed that one can ***change numeraire*** to any asset B and associate to it an equivalent probability measure under which A/B is a martingale for all other assets A. In some problems (such as certain Asian options or the passport option), it is advantageous to take the underlying asset itself as the numeraire.

3. THE PRINCIPLE OF NUMERAIRE INVARIANCE

We fix a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ with a finite time horizon $t \in [0, T]$. We denote the stochastic integral of a locally bounded predictable integrand $\theta = (\theta^1, \cdots, \theta^n)$ against a (vector) semimartingale $X = (X^1, \cdots, X^n)$ by

$$\theta \cdot X = \sum_{i=1}^{n} \int_0^{\cdot} \theta_t^i dX_t^i.$$

In what follows, A will denote a vector semimartingale:

$$A = (A^1, \cdots, A^m). \qquad (m \geq 2)$$

Each A^i represents the observable price process of a traded (or replicable) *zero-dividend* asset. When $A^m, A_-^m > 0$, we will set

$$X^i := \frac{A^i}{A^m},$$

and

$$X := (X^1, \cdots, X^n), \qquad n := m - 1.$$

3.1. Self-financing trading strategies (SFTS)

A SFTS δ for a semimartingale $A = (A^1, \cdots, A^m)$ is a locally bounded predictable process $\delta = (\delta^1, \cdots, \delta^m)$ such that

$$\sum_{i=1}^{m} \delta^i A^i = \sum_{i=1}^{m} \delta_0^i A_0^i + \delta \cdot A. \tag{1}$$

This is equivalent to saying that $C = C_0 + \delta \cdot A$, i.e.,

$$dC = \sum_{i=1}^{m} \delta^i dA^i, \tag{2}$$

where C is the SFTS *price process* defined by

$$C := \sum_{i=1}^{m} \delta^i A^i. \tag{3}$$

Clearly, C is then a semimartingale, $\Delta C = \sum_i \delta^i \Delta A_i$, and thus

$$C_- = \sum_{i=1}^{m} \delta^i A^i_-.$$

The hedge ratio δ^i_t is interpreted as the number of shares invested in asset A^i at time t.

3.2. Numeraire invariance

Theorem 3.1 *Let δ be a SFTS for A and S be any (scalar) semimartingale. Then δ is also a SFTS for $SA = (SA^1, \cdots, SA^m)$, i.e., (with $C := \sum_1^m \delta^i A^i$),*

$$d(SC) = \sum_{i=1}^{m} \delta^i d(SA^i). \tag{4}$$

Proof. By Itô's product rule, then substituting for dC and C_- and regrouping, followed by Itô's product rule again,

$$\begin{aligned} d(SC) &= S_- dC + C_- dS + d[S, C] \\ &= S_- \sum_{i=1}^{m} \delta^i dA^i + \sum_{i=1}^{m} \delta^i A^i_- dS + \sum_{i=1}^{m} \delta^i d[S, A^i] \\ &= \sum_{i=1}^{m} \delta^i (S_- dA^i + A^i_- dS + d[S, A^i]) = \sum_{i=1}^{m} \delta^i d(SA^i). \end{aligned}$$

■

To our best knowledge, this result first appeared in the 1992 edition of Duffie (2001), where it is called the ***numeraire invariance theorem***. Duffie gives the same proof, but assumes that the A^i are (continuous) Itô processes. The only difference in the general case here is the use of left limits, primarily, substituting $C_- = \sum_1^m \delta^i A^i_-$ instead of $C = \sum_1^m \delta^i A^i$.

Interpreting S as an exchange rate, numeraire invariance means that the self-financing property is independent of the choice of base currency, which is intuitively obvious.

If $S, S_- > 0$, then $1/S$ is also a semimartingale. The result applied to $1/S$ implies that: *δ is a SFTS for A if and only if it is one for SA.* Thus, *if (3) holds then (2) and (4) are equivalent.*

3.3. Taking an asset as numeraire

Assume now $A^m, A^m_- > 0$, and apply the result to $S = 1/A^m$. It follows that:

Theorem 3.2 *δ is a SFTS for A if and only if it is a SFTS for $A/A^m = (X, 1)$, i.e., if and only if $F := C/A^m$ satisfies $F = F_0 + \delta' \cdot X$ where $\delta' := (\delta^1, \cdots, \delta^n)$. Clearly then*

$$\delta^m = F - \sum_{i=1}^n \delta^i X^i = F_- - \sum_{i=1}^n \delta^i X^i_- \qquad (F := \frac{C}{A^m}).$$

Conversely, given $\delta' = (\delta^1, \cdots, \delta^n)$ and an F_0, then with δ^m as above, $\delta = (\delta', \delta^m)$ is a SFTS for $(X, 1)$ with price process $F := F_0 + \delta' \cdot X$. Hence by numeraire invariance δ is a SFTS for A with price process $C = A^m F$. Numeraire invariance thus reduces dimensionality by one:

Theorem 3.3 *In order to find a SFTS δ with a given time-T payoff C_T, it is sufficient to find a process δ' and an F_0 such that $F_T = C_T/A^m_T$, where $F = F_0 + \delta' \cdot X$, or equivalently to find a process F such that $F_T = C_T/A^m_T$ and $dF = \sum_1^n \delta^i dX^i$ for some $\delta^1, \cdots, \delta^n$.*

Since $\delta^m = F - \sum_{i=1}^n \delta^i X^i$, the m-th delta δ^m is like F determined by δ' and F_0. As such, one interprets the m-th asset as the ***numeraire asset*** chosen to finance an otherwise arbitrary trading strategy δ' in the other assets, post an initial investment of $C_0 = A^m_0 F_0$.

3.4. Application to unique pricing

One calls A ***arbitrage free*** if there exists a *state price density*, i.e., semimartingale S such that $S, S_- > 0$ and SA^i are martingales for all i.

The (bounded) ***law of one price*** then holds:

Theorem 3.4 *If A is arbitrage free and δ is a* ***bounded*** *SFTS for A then SC is a martingale where $C := \sum_{i=1}^m \delta^i A^i$; consequently $C = 0$ if $C_T = 0$.*

Proof. By *numeraire invariance*, $d(SC) = \sum_{i=1}^m \delta^i d(SA^i)$. Thus SC is a local martingale. But since δ is bounded, SC is dominated by a martingale. So SC is a martingale. ■

By a simple and well-known argument:

Theorem 3.5 *If $A^m, A^m_- > 0$, then A is arbitrage free if and only if there exists an equivalent probability measure $\mathbb{Q}$ such that A^i/A^m are $\mathbb{Q}$-martingales, for all i.*

The equivalent martingale measure $\mathbb{Q}$ is related to S by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{S_T A^m_T}{\mathbb{E}[S_0 A^m_0]}.$$

Theorem 3.6 *If δ is a* ***bounded*** *SFTS, then C/A^m is a $\mathbb{Q}$-martingale, where $C := \sum_{i=1}^m \delta^i A^i$; hence*

$$C_t = A^m_t \mathbb{E}^{\mathbb{Q}}[\frac{C_T}{A^m_T} \mid \mathcal{F}_t].$$

Proof. By *numeraire invariance*, $d(C/A^m) = \sum_{i=1}^m \delta^i d(A^i/A^m)$. So C/A^m is a local martingale. Since δ is bounded, C/A^m is dominated by a martingale. So C/A^m is a martingale. ■

3.5. Unique hedging

Let A be arbitrage-free and δ be a bounded SFTS for A. Then, as before, $X^i := A^i/A^m$ and $F := C/A^m$ are $\mathbb{Q}$-martingales, and $dF = \sum_{i=1}^n \delta^i dX^i$ by numeraire invariance. Assume that X^i are $\mathbb{Q}$-locally square-integrable (e.g., continuous). Then, $d\langle F\rangle^{\mathbb{Q}} = \sum_{ij=1}^n \delta^i\delta^j d\langle X^i, X^j\rangle^{\mathbb{Q}}$. (Here, $\langle X^i, X^j\rangle^{\mathbb{Q}}$ is the $\mathbb{Q}$-compensator of $[X^i, X^j]$; so it equals the latter in the continuous case.) Clearly, $\langle F\rangle^{\mathbb{Q}} = 0$ if $F_T = 0$. Thus:

Theorem 3.7 *If $\langle X^i\rangle^{\mathbb{Q}}$ are absolutely continuous and the $n \times n$ matrix $(\frac{d}{dt}\langle X^i, X^j\rangle^{\mathbb{Q}})$ is **nonsingular**, then given any random variable R, there exists **at most one bounded SFTS** δ for A with $\sum_{i=1}^m \delta_T^i A_T^i = R$.*

When there are "redundant assets", the matrix is singular, and replication is *not* unique.

4. APPLICATION TO DIFFUSION PROCESSES

4.1. Pricing and hedging

Let $A = (A^1, \cdots, A^m)$ be a semimartingale with $A, A_- > 0$ such that the price ratios $X^i := A^i/A^m$ follow the SDE system

$$dX_t^i = X_t^i \sum_{j=1}^k \varphi_{ij}(t, X_t)(dZ_t^j + \phi^j dt), \qquad (i = 1, \cdots, n := m-1)$$

where Z^j are independent Brownian motions, $\varphi_{ij}(t,x)$ are *bounded* continuous functions, and $\mathbb{E}\, e^{\frac{1}{2}\sum_j \int_0^T (\phi_t^j)^2 dt} < \infty$. (Note, we allow A^i be discontinuous.) Define the martingale

$$M := e^{-\sum_{j=1}^k (\int \phi^j dZ^j + \frac{1}{2}\int (\phi^i)^2 dt)},$$

and the measure $\mathbb{Q}$ by $d\mathbb{Q} = M_T d\mathbb{P}$. Then $W^j := Z^j + \int \phi^j dt$ are $\mathbb{Q}$-Brownian motions and are $\mathbb{Q}$-independent since $[W^j, W^k] = 0$ for $j \neq k$. The X^i are $\mathbb{Q}$-martingales since

$$dX_t^i = X_t^i \sum_{j=1}^k \varphi_{ij}(t, X_t) dW_t^j, \tag{5}$$

and $\varphi_{ij}(t,x)$ are bounded. *Thus A is **arbitrage-free**.*

For each $s \le T$ and $x \in \mathbb{R}_+^n$, there is a unique continuous positive $\mathbb{Q}$-square-integrable martingale $X^{s,x} = (X_t^{s,x})$ on $[s,T]$ with $X_s^{s,x} = x$ satisfying this SDE, and we have $X = X^{0,X_0}$.

Now, let $h(a)$, $a \in \mathbb{R}_+^m > 0$, be a homogeneous Borel function of linear growth. Define

$$g(x) := h(x, 1), \quad x \in \mathbb{R}_+^n.$$

Define the function $f(t,x)$ satisfying $f(T,x) = g(x)$ by,

$$f(t,x) := \mathbb{E}^{\mathbb{Q}}\, g(X_T^{t,x}). \tag{6}$$

(Intuitively, $f(t,x) = \mathbb{E}[g(X_T) \mid X_t = x]$.) Then the Markov property holds, i.e., we have,

$$F_t := f(t, X_t) = \mathbb{E}^{\mathbb{Q}}(g(X_T) \mid \mathcal{F}_t). \tag{7}$$

Thus $F = (f(t, X_t))$ is a $\mathbb{Q}$-martingale, and since X^i are too, assuming that $f(t,x)$ is $C^{1,2}$, Itô's formula yields (setting the martingale and drift parts equal),

$$dF_t = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(t, X_t) dX_t^i, \tag{8}$$

and

$$\frac{\partial f}{\partial t}(t, X_t)dt + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(t, X_t) d[X^i, X^j]_t = 0. \tag{9}$$

By (8) and ***numeraire invariance***, δ is a SFTS for A, where

$$\delta_t^i := \frac{\partial f}{\partial x_i}(t, X_t),\ i \leq n, \qquad \delta^m := F - \sum_{i=1}^{n} \delta^i X_i. \tag{10}$$

Clearly, the price process of this SFTS is $C = A^m F$ (by the definition of δ^m). Moreover, $C_T = h(A_T)$ since $F_T = g(X_T)$ and $h(a)$ is homogeneous.

By (9), on the support X, $f(t,x)$ satisfies the PDE

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sum_{i,j=1}^{n} x_i x_j \sigma_{ij}(t,x) \frac{\partial^2 f}{\partial x_i \partial x_j} = 0, \tag{11}$$

where

$$\sigma_{ij}(t,x) := \sum_{l=1}^{k} \varphi_{il}(t,x)\varphi_{jl}(t,x).$$

By the invariance of Itô's formula under the change of coordinates, the change of variable $L^i = \frac{X^i}{X^{i+1}} - 1$ $(i < n)$, $L^n = X^n - 1$, transforms (11) into the *Libor market model* PDE.

4.2. The homogeneous solution

The option price process and the deltas are already found, but let us also discuss the homogeneous option price function defined by

$$c(t,a) := a_m f(t, \frac{a_1}{a_m}, \cdots, \frac{a_n}{a_m}).$$

Then $C_t = c(t, A_t)$. Agreeably, $\delta_t^i = \frac{\partial c}{\partial a_i}(t, A_t)$ by (10). (For $i = m$ use Euler's formula for $c(t,a)$). By the continuity of X and (10), $\delta_t^i = \frac{\partial c}{\partial a_i}(t, A_{t\_})$ too. Therefore by Itô's formula,

$$\frac{\partial c}{\partial t}(t, A_{t\_})dt + \frac{1}{2}\sum_{i,j=1}^{m} \frac{\partial^2 c}{\partial a_i \partial a_j}(t, A_{t\_}) d[A^i, A^j]_t^c = 0. \tag{12}$$

(The sum of jumps term in Itô's formula drops out since $\Delta C = \sum \delta^i \Delta A^i$.) This yields the PDE $\frac{\partial c}{\partial t} + \frac{1}{2}\sum_{i,j} a_i a_j \sigma_{ij}^A(t,a) \frac{\partial^2 c}{\partial a_i \partial a_j} = 0$ for the special case $d[A^i, A^j]_t = A_t^i A_t^j \sigma_{ij}^A(t, A_t)dt$ for some functions $\sigma_{ij}^A(t,a)$. The quotient-space PDE (11) is more fundamental for it holds in general (even when A is not a diffusion or is discontinuous) and has one lower dimension.

4.3. Deterministic volatility case

Assume φ_{ij}, and hence σ_{ij}, are independent of x. Then we simply have $X_T^{t,x} = xX_T/X_t$. Hence by (6),

$$f(t,x) := \mathbb{E}^{\mathbb{Q}}[g(x_1\frac{X_T^1}{X_t^1}, \cdots, x_n\frac{X_T^n}{X_t^n})]. \tag{13}$$

Conditioned on $\mathcal{F}_t$ and unconditionally, X_T/X_t is $\mathbb{Q}$-*multivariately lognormally distributed*, with mean $(1, \cdots, 1)$ and log-covariances $\int_t^T \sigma_{ij}(s)ds$. Let $P(t,T,z)$, denote its distribution function. Then by (13), we obtain

$$f(t,x) = \int_{\mathbb{R}_+^n} g(x_1z_1, \cdots, x_nz_n)P(t,T,dz). \tag{14}$$

If $\partial g/\partial x_i$ and $g(x) - \sum x_i \partial g/\partial x_i$ are bounded, then so is δ, since

$$\frac{\partial f}{\partial x_i}(t,x) = \mathbb{E}^{\mathbb{Q}}[\frac{X_T^i}{X_t^i}\frac{\partial g}{\partial x_i}(x_1\frac{X_T^1}{X_t^1}, \cdots, x_n\frac{X_T^n}{X_t^n})].$$

5. APPLICATION TO EXPONENTIAL POISSON MODEL

5.1. Option to exchange two assets

Let A and B denote the asset price processes. Assume $A = BX$, where

$$X_t = X_0 e^{\beta P_t - (e^\beta - 1)\lambda t} \tag{15}$$

for some constants $\beta \neq 0$, $\lambda > 0$ and semimartingale P such that $[P] = P$ and $P_0 = 0$ (so, $P_t = \sum_{s \le t} 1_{\Delta P_s \neq 0}$), e.g., a Poisson (or Cox) process. Equivalently, by Itô's formula, X follows

$$dX_t = X_{t-}(e^\beta - 1)d(P_t - \lambda t). \tag{16}$$

Define the function $f(t,x)$, $x > 0$ by

$$f(t,x) := \sum_{n=0}^{\infty} (xe^{\beta n - (e^\beta - 1)\lambda(T-t)} - 1)^+ \frac{\lambda^n}{n!}(T-t)^n e^{-\lambda(T-t)}, \tag{17}$$

Clearly $f(T,x) = (x-1)^+$. Define $u(t,p) := f(t, X_0 e^{\beta p - (e^\beta - 1)\lambda t})$. One directly verifies that

$$\frac{\partial u}{\partial t}(t,p) + \lambda(u(t,p+1) - u(t,p)) = 0,$$

Using this, one can show that

$$dF = \delta^A dX, \qquad F_t := f(t, X_t), \tag{18}$$

where,

$$\delta_t^A := \delta_A(t, X_{t-}), \quad \delta_A(t,x) := \frac{f(t, e^\beta x) - f(t,x)}{(e^\beta - 1)x}.$$

Thus by ***numeraire invariance*** (δ^A, δ^B) is a SFTS for A with price process $C = BF$, where

$$\delta^B := F_- - \delta^A X_- = F - \delta^A X.$$

Further, $C_T = (A_T - B_T)^+$ since $F_T = (X_T - 1)^+$.

Also, this is a *bounded* SFTS. In fact, $0 \leq \delta^A \leq 1$ and $-1 \leq \delta^B \leq 0$.

5.2. Multivariate exponential Poisson model

Let $A > 0$ be an m-dimensional semimartingale with $A_- > 0$. Set $X := (A^i/A^m)_{i=1}^n$, $n := m-1$. Assume

$$X_t^i := X_0^i \exp(\sum_{j=1}^k (\beta_{ij} P_t^j - (e^{\beta_{ij}} - 1)\lambda_j t)),$$

$(1 \leq k \leq n)$ or equivalently,

$$dX_t^i = X_{t-}^i \sum_{j=1}^k (e^{\beta_{ij}} - 1)(dP_t^j - \lambda_j dt),$$

where, β_{ij} are constants with the $n \times k$ matrix $(e^{\beta_{ij}} - 1)$ of full rank, $\lambda_j > 0$ are constants, and P^j are semimartingales such that $[P^j] = P^j$, $P_0^j = 0$ and $[P^j, P^l] = 0$ for $j \neq l$.

Let $h(a)$, $a \in \mathbb{R}_+^m$ be a given payoff function, assumed ***homogeneous*** of degree 1 and of linear growth in a. Define

$$g(x) := h(x,1), \quad x \in \mathbb{R}_+^n, \ \ n := m-1.$$

Define

$$f(t,x) := \sum_{q_1,\cdots,q_n=0}^{\infty} g(x_1 e^{\sum_{j=1}^n (\beta_{1j} q_j - (e^{\beta_{1j}} - 1)\lambda_j (T-t))}, \cdots,$$

$$x_n e^{\sum_{j=1}^n (\beta_{nj} q_j - (e^{\beta_{nj}} - 1)\lambda_j (T-t))}) \prod_{i=1}^n \frac{\lambda_i^{q_i}}{q_i!} (T-t)^{q_i} e^{-\lambda_i (T-t)}.$$

Let $\alpha = (\alpha_{ij})$ be any $n \times k$ matrix such that for $1 \leq j, l \leq k$, $\sum_{i=1}^n (e^{\beta_{il}} - 1)\alpha_{ij} = 1$ if $j = l$ and 0 otherwise. Define

$$\delta_t^i := \delta_i(t, X_{t-}), \quad (1 \leq i \leq n) \tag{19}$$

where

$$\delta_i(t,x) := \frac{1}{x_i} \sum_{j=1}^k \alpha_{ij} (f(t, e^{\beta_{1j}} x_1, \cdots, e^{\beta_{nj}} x_n) - f(t,x)).$$

Then one can show

$$dF = \sum_{i=1}^n \delta^i dX^i, \quad F_t := f(t, X_t). \tag{20}$$

Hence by ***numeraire invariance***, $\delta = (\delta^1, \cdots, \delta^n, \delta^m)$ is a SFTS for A, where $\delta^m := F - \sum_{i=1}^n \delta^i X^i$. Its price process $C = \sum_1^m \delta^i A^i = C_0 + \delta \cdot A$ is clearly given by $A^m F$:

$$C_t = A_t^m f(t, X_t). \tag{21}$$

Further, $C_T = h(A_T)$ because $h(a)$ is homogeneous of degree 1 and $f(T, x) = g(x) := h(x, 1)$.

Moreover, δ^i are bounded if $\gamma_i(x)$ are bounded, where $\gamma_m(x) := g(x) - \sum_{i=1}^n \gamma_i(x) x_i$ and

$$\gamma_i(x) := \frac{1}{x_i} \sum_{j=1}^k \alpha_{ij} (g(e^{\beta_{1j}} x_1, \cdots, e^{\beta_{nj}} x_n) - g(x)) \qquad (i \leq n).$$

5.3. Relation to Poisson predictable representation

Let $P = (P^1, \cdots, P^k)$ be a vector of independent Poisson processes P^i with intensities $\lambda_i > 0$. Let $v(p)$, $p \in \mathbb{R}^k$, be a function of exponential linear growth. Then, one has the following representation:

$$v(P_T) = \sum_{q_1, \cdots, q_k = 0}^{\infty} v(q_1, \cdots, q_k) \prod_{i=1}^k \frac{\lambda_i^{q_i}}{q_i!} T^{q_i} e^{-\lambda_i T} + \sum_{i=1}^k \int_0^T \Delta_i u(t, P_{t-}) d(P_t^i - \lambda_i t),$$

where $\Delta_i u(t, p) := u(t, p_1, \cdots, p_i + 1, \cdots p_n) - u(t, p))$ and

$$u(t, p) := \sum_{q_1, \cdots, q_k = 0}^{\infty} v(p + q) \prod_{i=1}^k \frac{\lambda_i^{q_i}}{q_i!} (T - t)^{q_i} e^{-\lambda_i (T - t)}.$$

Also, $u(t, p)$ satisfies the partial difference equation and the SDE

$$\frac{\partial u}{\partial t}(t, p) + \sum_{i=1}^k \lambda_i \Delta_i u(t, p) = 0;$$

$$du(t, P_t) = \sum_{i=1}^k \Delta_i u(t, P_-) d(P^i - \lambda_i t).$$

6. MISCELLANEOUS CONSIDERATIONS

6.1. The role of homogeneity

Let A be continuous semimartingale and δ be a SFTS for A. Assume $C_t = c(t, A_t)$ for a $C^{1,2}$ function $c(t, a)$. Since $dC = \sum \delta^i dA^i$, by Itô's formula,

$$\frac{\partial c}{\partial t}(t, A_t) dt + \frac{1}{2} \sum_{i,j=1}^m \frac{\partial^2 c}{\partial a_i \partial a_j}(t, A_t) d[A^i, A^j]_t = \sum_{i=1}^m (\delta_t^i - \frac{\partial c}{\partial a_i}(t, A_t)) dA_t^i. \tag{22}$$

In general, $\sum_{i,j}(\delta^i - \frac{\partial c}{\partial a_i})(\delta^j - \frac{\partial c}{\partial a_j})d[A^i, A^j] = 0$ since the (left so) right hand side of (22) has finite variation and hence zero quadratic variation. Thus, if $[A^i]$ are absolutely continuous and the $m \times m$ matrix $(\frac{d}{dt}[A^i, A^j])$ is *nonsingular*, then $\delta^i_t = \frac{\partial c}{\partial a_i}(t, A_t)$, and so by (22),

$$\frac{\partial c}{\partial t}(t, A_t)dt + \frac{1}{2}\sum_{i,j=1}^{m} \frac{\partial^2 c}{\partial a_i \partial a_j}(t, A_t)d[A^i, A^j]_t = 0. \tag{23}$$

Moreover, since $C = \sum_i \delta^i A^i$, we then have $c(t, A_t) = \sum_i \frac{\partial c}{\partial a_i}(t, A_t)A^i_t$. So, if the support of A_t is a cone, then it follows that $c(t, a)$ is necessarily *homogeneous of degree 1* in a on that cone. Consequently, only homogeneous payoffs can be so replicated in this nonsingular case.

In some singular cases, e.g., the Black-Scholes or Markovian short-rate models, there also exist infinitely many *nonhomogeneous* functions $c(t, a)$ satisfying $C_t = c(t, A_t)$. This is simply because for each t the support of A_t is a proper surface in $\mathbb{R}^m$ in these models, and obviously there exist infinitely many distinct functions on $\mathbb{R}^m$ that coincide on any surface.

Assume $M^i := e^{-\int_0^\cdot r_t dt} A^i$ are local martingales under an equivalent measure for some predictable process r. Then $dA^i = rA^i dt + e^{\int r dt} dM^i$. Thus, using $C = \sum_i \delta^i A^i$ and (22),

$$\frac{\partial c}{\partial t}(t, A_t)dt + \frac{1}{2}\sum_{i,j=1}^{m} \frac{\partial^2 c}{\partial a_i \partial a_j}(t, A_t)d[A^i, A^j]_t = r_t(C_t - \sum_{i=1}^{m} \frac{\partial c}{\partial a_i}(t, A_t)A^i_t)dt. \tag{24}$$

This "PDE" is valid also for nonhomogeneous functions. It is the type of PDE encountered in the Black-Scholes or Markovian short-rate models. Of course, if we choose $c(t, a)$ to be homogeneous — which we can thanks to numeraire invariance — then it simplifies to (23).

6.2. Extension to dividends

Consider m assets with positive price processes $\hat{A}^i$ and continuous dividend yields y^i_t. When there exist traded or replicable zero-dividend assets A^i such that $A^i_T = \hat{A}^i_T$, the problem reduces to pricing and hedging (European) options on the A^i.

All that is required is that the $2m$ assets A^i and $\tilde{A}^i$ be arbitrage free, where

$$\tilde{A}^i_t := e^{\int_0^t y^i_s ds} \hat{A}^i_t$$

is the price of the zero-dividend asset that initially buys one share of $\hat{A}$ and thereon continually reinvests all dividends in $\hat{A}$ itself. (When y^i is deterministic, this requires $A^i_t = e^{-\int_t^T y^i_s ds} \hat{A}^i_t$.)

For instance, consider an exchange option ($m = 2$). Say $\hat{A}$ and $\hat{B}$ are the yen/dollar and yen/Euro exchange rates viewed as yen-denominated dividend assets. Then A is the yen-value of the U.S. T-maturity zero-coupon bond and $\tilde{A}$ is the yen-value of the U.S. money market asset. This exchange option is equivalent to a Euro-denominated call struck at 1 on the Euro/dollar exchange rate $\hat{A}/\hat{B}$. The ratio A/B is the *forward* Euro/dollar exchange rate. If it has deterministic volatility, we are as in a setting of Jamshidian (1993) with results similar to next section.

6.3. Change of numeraire

For the exchange option, one has to calculate $\mathbb{E}\,(X - Y)^+$ for certain integrable random variables X and $Y > 0$. Such expectations often become more tractable by a change of measure as in El-Karoui et al. (1995). Define the equivalent probability measure $\mathbb{Q}$ by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} := \frac{Y}{\mathbb{E}(Y)}.$$

Clearly,

$$\mathbb{E}^{\mathbb{Q}}(\frac{X}{Y}) = \frac{\mathbb{E}(X)}{\mathbb{E}(Y)}. \tag{25}$$

Replacing X by $(X - Y)^+$ in (25) and using the homogeneity to factor out Y,

$$\mathbb{E}\,(X - Y)^+ = \mathbb{E}(Y)\mathbb{E}^{\mathbb{Q}}(\frac{X}{Y} - 1)^+. \tag{26}$$

If X/Y is $\mathbb{Q}$-lognormally distributed then (25) and (26) readily yield,

$$\mathbb{E}\,(X - Y)^+ = \mathbb{E}(X)N(\frac{\log(\mathbb{E}X/\mathbb{E}Y)}{\sqrt{\nu^{\mathbb{Q}}}} + \frac{\sqrt{\nu^{\mathbb{Q}}}}{2}) - \mathbb{E}(Y)N(\frac{\log(\mathbb{E}X/\mathbb{E}Y)}{\sqrt{\nu^{\mathbb{Q}}}} - \frac{\sqrt{\nu^{\mathbb{Q}}}}{2}), \tag{27}$$

where $\nu^{\mathbb{Q}} := \text{var}^{\mathbb{Q}}[\log(X/Y)]$ and $N(\cdot)$ denotes standard the normal distribution function.

If X and Y are bivariately lognormally distributed, as in Merton's and Margrabe's models, then it is not difficult to show that X/Y is lognormally distributed in both $\mathbb{P}$ and $\mathbb{Q}$ with the same log-variance $\nu^{\mathbb{Q}} = \nu := \text{var}[\log(X/Y)]$. Then $\nu^{\mathbb{Q}}$ can be replaced with ν in (27).

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CONTRIBUTED TALKS

PRICING CATASTROPHE OPTIONS IN INCOMPLETE MARKETS

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Abstract

In complete markets, pricing financial products is easy (at least from a theoretical point of view). In incomplete markets (e.g. when the underlying process has jumps with random size, such as an insurance loss process), the price is no longer unique. So on the one hand, it becomes difficult to provide a tractable price of insurance-linked derivatives. On the other hand, when facing catastrophic losses, using the pure premium as a price might not be relevant (e.g. for solvency issues). Both financial market and (re)insurance industry have proposed techniques to price identical hedging products that can be related (e.g. Esscher transform and more generally distorted risk measures in insurance, Gerber-Shiu transform in finance). In this paper, we focus on indifference utility techniques, assuming that stock prices have jumps, related to major catastrophic losses, and thus, partial hedging should then be possible.

1. INTRODUCTION AND MOTIVATIONS

1.1. Notations and definition

The buyer of an insurance contract buys the right to get reimbursed – by the insurance company – all the losses which occurred during a given period of time, (for which the loss amount exceeded a deductible, if any). The buyer of a call option buys the right to buy the underlying stock from the seller to capture its increased value above the strike price.

Both (financial) options and insurance policies have the objective to transfer a risk from one part to another, against a specific payment (called premium in insurance). But classical techniques in insurance (based on the use of the pure premium, $\mathbb{E}_{\mathbb{P}}[(X-d)_+]$) and finance (based on the assumption of complete market and no-arbitrage, so that the price of a call option is $\mathbb{E}_{\mathbb{Q}}[(X-K)_+]$) are no longer relevant.

On the one hand, most of the techniques designed to price insurance contracts have been developed for standard risks, not to hedge against catastrophes. Pricing reinsurance, where events are rare and with high severity, is more challenging, and the use of the pure premium might not be relevant, for solvency issues. On the other hand, the closed-form model for pricing financial options obtained in the beginning of the 70's, assumed that a volatility of the underlying stock was available, known and constant, and that the underlying price was continuous. Those two assumptions were related to the idea of *complete* markets.

The challenge in insurance-linked derivatives is to find a price for those financial products, and to relate them to classical insurance covers, since the question asked by any risk manager is "*which risk transfer technique is the cheapest one* ?". But, as mentioned in Finn and Lane (1997), one has to keep in mind, "*there are no right prices of insurance, there is simply the transacted market price which is high enough to bring forth sellers and low enough to induce buyers*". From a terminology point of view, Holtan (2007) suggested to distinguish the *price* of an option or the *premium* of an insurance contract, and the so-called *value* of those products. The difference depends mostly on market conditions.

1.2. Trading insurance risks

Insurance risks are traded as long as there are insurance contracts, buyers and sellers, but they are traded within the (re)insurance market *only*. To compare with the financial market, derivatives are traded on a structured market, as well as the underlying stock, which will make replication possible (and therefore hedging and pricing derivatives). In the case of insurance risks, we can image that some standard contracts could be – somehow – concluded with financial companies, but the underlying risk (cumulated insurance claims for indemnity covers, or weather related index) is not traded on financial market: in that case, there is few chance that insurance risks could be replicated, and therefore classical techniques to price are no longer valid.

Assuming that financial markets integrate information about catastrophes (and more generally any insurance related information), it might be possible to hedge insurance risks on financial markets. But most of the assumptions underlying the Black & Scholes assumptions are usually not fulfilled with insurance-linked derivatives

- the market is not complete, and catastrophe (or mortality risk) cannot be replicated,
- the guarantees are not actively traded, and thus, it is difficult to assume no-arbitrage,
- the hedging portfolio should be continuously rebalanced, and there should be large transaction costs,
- if the portfolio is not continuously rebalanced, we introduce an hedging error,
- equity prices are not driven by a geometric Brownian motion process.

The goal of this paper is to focus on catastrophe options and to find *a* price for those financial products.

1.3. Outline of the paper

In Section 2, classical results on financial pricing will be recalled, focusing on assumptions underlying the *fundamental theorem of asset pricing*. Then, in Section 3, classical insurance pricing

methods will be presented (based either on expected utility principles or using distorted risk measures). In Section 4, classical financial methods to avoid drawbacks of incompleteness will be presented (and related to insurance pricing), to find *a* possible martingale measure.
And finally, in Section 5, we will study a model based on indifference utility pricing. The underlying idea is that financial markets can be affected by shocks related to major insurance losses. As mentioned in Shimpi (1997) with a qualitative point of view, "*from an insurance industry perspective, the closer the index is to the loss experience, the better the ability to hedge the loss exposure of insurers*". Even, if a stock price is not perfectly correlated with insurance losses, at least its discontinuous part can be. The goal here will be to see if those jumps in financial prices can be used to hedge against catastrophes.

2. PRICING FINANCIAL PRODUCTS IN COMPLETE MARKETS

Harrison and Pliska (1981) said that a market is *complete* if there is only one equivalent martingale measure to the underlying stock price. Insurance markets would be complete if there would be a unique price for each risk, and if each contract could perfectly be hedged in the market. As mentioned in Embrechts and Meister (1997) market incompleteness can be explained by jumps in the underlying stochastic process, with random size, by stochastic volatility, or by the existence of transaction costs (or more generally any *friction*). Hence, in complete markets, all relevant market information is supposed to be known and integrated in the price: no investor will expect a higher return than the risk free rate of return. The technique is to tune the historical probability $\mathbb{P}$ into an equivalent probability measure $\mathbb{Q}$, so that the price process of the underlying financial asset becomes a martingale under probability $\mathbb{Q}$, i.e. $\mathbb{E}_{\mathbb{Q}}(S_{t+h}|\mathcal{F}_t) = S_t$. Hence, it becomes impossible to use the history of the stock to earn money: all possible relevant information is already included in its spot price. This link between no-arbitrage assumption and martingale processes is the *fundamental theorem of asset pricing* (see Delbaen and Schachermayer (1994)): the price of a contingent claim X (e.g. the payoff of the European call with strike K and maturity T is $X = (S_T - K)_+$) is $\pi(X) = \mathbb{E}_{\mathbb{Q}}(e^{-rT}X)$, assuming constant risk free rate r, where $\mathbb{Q}$ stands for *the* risk neutral probability measure equivalent to $\mathbb{P}$.
The Black & Scholes model assumes that the price of a risky asset $(S_t)_{t\geq 0}$ satisfies $S_t = S_0 \exp(X_t)$ where $(X_t)_{t\geq 0}$ is a Brownian process, i.e. $S_t = S_0 \exp(\mu + \sigma X_t^0)$ where $(X_t^0)_{t\geq 0}$ is a standard Brownian motion. Having a geometric Brownian motion reflecting uncertainty on financial markets (for stock prices (S_t)) yield simple and nice pricing formulas. The most difficult practical issue is that the only unknown valuation parameter is the stock volatility σ, making option dealers simply "*volatility dealers*": the value of a financial option depends on the volatility of the underlying financial stock, and not its expected return (which has to be equal to the risk free rate of return), leading to a "*risk neutral*" pricing.

3. PRICING INSURANCE PRODUCTS

The basic principle of insurance is the law of large numbers: if the premium asked is $\mathbb{E}_{\mathbb{P}}(X)$, then the insurance company makes a null profit, *on average*. Feller (1945) called $\mathbb{E}_{\mathbb{P}}(X)$ the *fair* price (of a game, in his terminology). In the terms of d'Alembert, the pure premium is the "*inner product of probabilities and losses*". Thus, $\mathbb{E}_{\mathbb{P}}(X)$ is called pure premium, but using it as the price of a risk, the company is very likely to lose money (since the balance is only *on average*). Therefore, traditional premium calculation principles are

$$\begin{cases} \pi(X) = \mathbb{E}_{\mathbb{P}}(X)\text{: equivalence principle (pure premium)} \\ \pi(X) = \mathbb{E}_{\mathbb{P}}(X) + \lambda\mathbb{E}_{\mathbb{P}}(X)\text{: expected value principle} \\ \pi(X) = \mathbb{E}_{\mathbb{P}}(X) + \lambda\text{Var}_{\mathbb{P}}(X)\text{: variance principle} \\ \pi(X) = \mathbb{E}_{\mathbb{P}}(X) + \lambda\sqrt{\text{Var}_{\mathbb{P}}(X)}\text{: standard-deviation principle} \end{cases}$$

Remark 3.1 *For the standard-deviation principle, if X has a Gaussian distribution, then $\pi(X)$ is simply a quantile of X.*

3.1. Pricing using expected utility principles

The fact that the pure premium might not be appropriate has been mentionned starting from Saint-Petersburg's paradox. One of the answers was to introduce a *moral utility* of X. A utility function U is an increasing twice differentiable function on $\mathbb{R}$, strictly increasing ($U'(\cdot) > 0$, i.e. "*more is better*") and concave ($U''(\cdot) < 0$, i.e. "*marginal utility decreases*"). Concavity is related to risk aversion; since we assume that the agent is willing to transfer a risk, it is relevant to assume that U is concave.

Example 3.1 *Three types of expected utility are frequently used in the context of expected utility,*

$$\begin{cases} \forall x \in \mathbb{R}_+^\star, U_L(x) = \log(x)\text{: logarithmic utility} \\ \forall x \in \mathbb{R}_+^\star, U_P(x) = \dfrac{x^p}{p} \text{ where } p \in]-\infty, 0[\cup]0,1[\text{: power utility} \\ \forall x \in \mathbb{R}, U_E(x) = -\exp(-\dfrac{x}{x_0})\text{: exponential utility.} \end{cases}$$

Functions U_P and U_L belong to the set of functions having constant relative risk aversion (CRRA). Functions U_E belong to the set of functions having constant absolute risk aversion (CARA).

Given utility function U, the premium π that an agent is willing to pay to transfer loss X is a solution to the following equation

$$U(\omega - \pi) = \mathbb{E}_{\mathbb{P}}(U(\omega - X)) \tag{1}$$

where ω denotes the initial wealth of the insured. Using Jensen's inequality (since U is assumed to be concave), note that $\pi \geq \mathbb{E}_{\mathbb{P}}(X)$.

Example 3.2 *Assuming exponential utility, i.e.* $U(x) = -e^{-x/x_0}$ *(with constant risk aversion* $1/x_0$*), then* $\pi = x_0 \log \mathbb{E}_{\mathbb{P}}(e^{X/x_0})$ *(also called entropy measure).*

Borch (1962) observed that the price of reinsurance contracts obtained with HARA utility functions (more general than CARA and CRRA) is quite similar to the financial formulas: "*This indicates that the theory of insurance premiums and the theory of asset prices are special cases of a more general theory*". This emphasizes the idea that it should be possible to relate insurance and finance valuation techniques.

3.2. Pricing using distorted risk measures

Using the duality principle (see Yaari (1987)), instead of distorting losses using a utility function, an alternative is to use a distortion of probabilities (leading to the *dual* approach, since the expected value can be seen as an inner product, as mentioned already by d'Alembert). Hence, the agent solves the dual version of Equation (1), i.e. (with an abuse of notation to highlight duality, see Remark 3.2)

$$\omega - \pi = \mathbb{E}_{g\circ\mathbb{P}}(\omega - X) = \int (\omega - x) g \circ \mathbb{P}(dx), \tag{2}$$

or equivalently, $\pi = \int x g \circ \mathbb{P}(dx) = \int g(\mathbb{P}(X > x))dx$ in the case X is a positive random variable, where g is a *distortion* measure, i.e. an increasing function on $[0,1]$, with $g(0) = 0$ and $g(1) = 1$.

Remark 3.2 *Note that this probability distortion does not necessarily define a probability measure, but only a* capacity*: if* $\mathbb{Q} = g \circ \mathbb{P}$, $\mathbb{Q}(\varnothing) = 0$ *(since* $g(0) = 0$*),* $\mathbb{Q}(\Omega) = 1$ *(since* $g(1) = 1$*), and* $\mathbb{Q}(A) \leq \mathbb{Q}(B)$ *if* $A \subset B$ *(since* g *is an increasing function). Hence, in Equation (2)* $\mathbb{E}_{g\circ\mathbb{P}}$ *is not an expected value, but a Choquet integral with respect to the (nonadditive) measure* $g \circ \mathbb{P}$.

Example 3.3 *If* $g(x) = \mathbf{1}(x > \alpha)$*, then* $\pi = F^{-1}(1-\alpha)$, $\alpha \in (0,1)$ *and* $F(x) = \mathbb{P}(X \leq x)$.

As a particular case of distorted probabilities, an important principle is the use of the Esscher transform,

$$\pi = \mathbb{E}_{\mathbb{Q}}(X) = \frac{\mathbb{E}_{\mathbb{P}}(X \cdot e^{\alpha X})}{\mathbb{E}_{\mathbb{P}}(e^{\alpha X})},$$

for some $\alpha > 0$. More generally, Delbaen and Haezendonck (1989) considered the following change of measure, so that the cumulative distribution function of the Radon-Nikodym derivative $d\mathbb{P}/d\mathbb{Q}$ is

$$G(x) = \frac{1}{\mathbb{E}_{\mathbb{P}}(e^{\beta(X)})} \int_0^x \exp(\beta(y))dF(y), \quad x \geq 0,$$

where F is the distribution function of X under $\mathbb{P}$, and $\beta(\cdot) : [0,\infty) \to (-\infty,\infty)$ satisfies $\mathbb{E}_{\mathbb{P}}(e^{\beta(X)}) < \infty$ and $\mathbb{E}_{\mathbb{P}}(Xe^{\beta(X)}) < \infty$.

Example 3.4 *If* $\beta(x) = \log(1 + b(x - \mathbb{E}_{\mathbb{P}}(X)))$*, then* $\pi = \mathbb{E}_{\mathbb{Q}}(X) = \mathbb{E}_{\mathbb{P}}(X) + b\mathit{Var}_{\mathbb{P}}(X)$*, which is the variance principle. If* $\beta(x) = \alpha x - \log \mathbb{E}_{\mathbb{P}}(e^{\alpha X})$*, for some* $\alpha > 0$*, then* $\pi = \dfrac{\mathbb{E}_{\mathbb{P}}(X \cdot e^{\alpha X})}{\mathbb{E}_{\mathbb{P}}(e^{\alpha X})}$.

4. PRICING FINANCIAL PRODUCTS IN INCOMPLETE MARKETS

4.1. A *natural* framework based on Lévy processes

As mentioned in Section 2, market incompleteness arises when the underlying stochastic process has jumps with random size. Hence, in a general framework, assume that the price of a risky asset $(S_t)_{t\geq 0}$ satisfies $S_t = S_0 \exp(X_t)$ where $(X_t)_{t\geq 0}$ is a Lévy process. Recall that $(X_t)_{t\geq 0}$ has independent, infinitely divisible and stationary increments, thus $X_{t+h} - X_t$ has characteristic function ϕ^h. The cumulant characteristic function satisfies the Lévy-Khintchine formula, i.e.

$$\psi(u) = \log \phi(u) = i\gamma u - \frac{1}{2}\sigma^2 u^2 + \int_{-\infty}^{+\infty} \left(e^{iux} - 1 - iux\mathbf{1}_{\{|x|<1\}}\right) \nu(dx),$$

where $\gamma \in \mathbb{R}$, $\sigma^2 \geq 0$ and ν is the so-called Lévy measure on $\mathbb{R} \setminus \{0\}$. Hence, the Lévy process is characterized either by ϕ (the characteristic function of X_1), or by the triplet (γ, σ^2, ν) in the Lévy-Khintchine formula.

Remark 4.1 *Again, except the case when $(X_t)_{t\geq 0}$ is a (pure) Poisson process or a Brownian motion, any Lévy model is an incomplete model.*

Market completeness is related to the existence of a *unique* martingale measure, also called the *predictable representation property* of a martingale: a martingale $(M_t)_{t\geq 0}$ satisfies this property if and only if for any square-integrable random variable $Z \in \mathcal{F}_T$, there exists a $\mathcal{F}_t$-predictable process $(a_t)_{t\in[0,T]}$ such that $Z = \mathbb{E}(Z) + \int_0^T a_s dM_s$. Actually, $(a_t)_{t\in[0,T]}$ is related to the self-balancing strategy. Nualart and Schoutens (2000) proved that under some weak assumptions, a Lévy process $(X_t)_{t\geq 0}$ can also have a *predictable representation property* of the form

$$Z = \mathbb{E}(Z) + \sum_{i=1}^{\infty} \int_0^T a_s^{(i)} d(H_s^{(i)} - \mathbb{E}(H_s^{(i)})),$$

where the $(a_t^{(i)})_{t\in[0,T]}$'s are $\mathcal{F}_t$-predictable processes, and where $(H_t^{(i)})_{t\in[0,T]}$ is the power jump process of order i, i.e. $H_t^{(1)} = X_t$ and $H_t^{(i)} = \sum_{0<s\leq t}[X_s - X_{s-}]^i$ for $i = 2, 3, \ldots$.As mentioned in Schoutens (2003), the predictable integrands $(a_t^{(i)})_{t\in[0,T]}$'s appearing in this representation can be interpreted in terms of minimal variance hedging strategies. Hence, those processes for $i = 2, 3, \ldots$ together correspond to the risk that cannot be hedged away. The term $(a_t^{(1)})_{t\geq 0}$ leads to the strategy that realizes the *closest* hedge to the claim.

A first idea, related to the classical pricing process in complete markets is to find an equivalent martingale measure, and to use it to derive a *price*. Hence, in Section 4.2 we will mention two ideas widely used to obtain one equivalent martingale measure $\mathbb{Q}$: one based on Gerber and Shiu (1994) (i.e. Esscher transform from insurance pricing) and the other one based on some *mean-correcting martingale measure*. The main problem in incomplete markets is that there is no replication portfolio. But it is still possible to super-replicate.

4.2. Finding one risk neutral measure

4.2.1. Using the Esscher transform

Following Gerber and Shiu (1994) we can – by using the Esscher transform – find in some cases at least one equivalent martingale measure $\mathbb{Q}$. More generally, Bühlmann et al. (1998) discussed the Esscher transform for specific classes of semi-martingales, with applications in finance and insurance.

Given a Lévy process $(X_t)_{t\geq 0}$ under $\mathbb{P}$ with characteristic function ϕ or triplet (γ, σ^2, ν), then under the Esscher transform probability measure $\mathbb{Q}_\alpha$ (as defined in Section 3.2), $(X_t)_{t\geq 0}$ is still a Lévy process with characteristic function ϕ_α such that

$$\log \phi_\alpha(u) = \log \phi(u - i\alpha) - \log \phi(-i\alpha),$$

and triplet $(\gamma_\alpha, \sigma_\alpha^2, \nu_\alpha)$ for X_1, where $\sigma_\alpha^2 = \sigma^2$, and

$$\gamma_\alpha = \gamma + \sigma^2\alpha + \int_{-1}^{+1} (e^{\alpha x} - 1)\nu(dx) \quad \text{and} \quad \nu_\alpha(dx) = e^{\alpha x}\nu(dx),$$

see e.g. Schoutens (2003).

Example 4.1 *A particular case is given when $(X_t)_{t\geq 0}$ is a Brownian motion under $\mathbb{P}$, then if $\alpha = (r - \mu)/\sigma^2$, $(X_t)_{t\geq 0}$ is still a Brownian motion under $\mathbb{Q}_\alpha$.*

Proposition 4.2 *If the price of a risky asset $(S_t)_{t\geq 0}$ satisfies $S_t = S_0 \exp(X_t)$, where $(X_t)_{t\geq 0}$ is a Lévy process, such that the random variable X_1 is non-degenerate and possesses a moment generating function $M(t) = \mathbb{E}_{\mathbb{P}}(e^{tX_1})$ on some interval (a, b), and if there exists $u \in (a, b - 1)$ such that $M(1 + u) = M(u)$, then $(e^{-rt}S_t)_{t\geq 0}$ is a $\mathbb{Q}_u$-martingale.*

Proof. See Shiryaev (1999). ∎

In order to have unicity, additional assumptions are necessary (see Kallsen and Shiryaev (2002)).

4.2.2. A mean-correcting martingale measure

Another way to obtain an equivalent martingale measure is inspired from the Black & Scholes model, and is related to some *mean-correcting martingale measure*. The underlying idea is to note that given a Lévy process $(X_t)_{t\geq 0}$ under $\mathbb{P}$ with characteristic function ϕ and triplet (γ, σ^2, ν), then the shifted process $(Y_t)_{t\geq 0} = (X_t - mt)_{t\geq 0}$ is also a Lévy process with characteristic function $\phi_m(u) = e^{ium}\phi(u)$ and triplet $(\gamma_m, \sigma_m^2, \nu_m) = (\gamma + m, \sigma^2, \nu)$ for X_1, see e.g. Schoutens (2003).

In the Black and Scholes model, we just switch from mean $\mu - \sigma^2/2$ to $r - \sigma^2/2$. In the Lévy model, the idea is to use the same kind of transform, $m_{\text{new}} = m_{\text{old}} + r - \log \phi(-i)$ (in the Black and Scholes model, $\log \phi(-i) = \alpha$). The choice of m_{new} will be such that the discounted price is a martingale.

5. THE INDIFFERENCE UTILITY APPROACH

As pointed out in Swiss Re (1999) about the pricing of financial stop loss contracts, "*the risk-neutral valuation technique traditionally used for the pricing of financial derivatives cannot be applied directly*". Nevertheless, practitioners need *a* price for insurance-linked derivatives.

Let $(S_t)_{t\geq 0}$ denote the accumulated insurance claim process, $S_t = \sum_{i=1}^{N_t} X_i$. Consider the classical stop-loss contract with $(S_T - K)_+$. The payoff of a call option is also $(S_T - K)_+$. Hence, those two covers are identical for an insurance company, willing to transfer risk claims exceeding priority K.

The idea of the pricing model here is to assume that the price of the financial asset has jumps related to the occurrence of catastrophes. This assumption can be validated by stylized facts, e.g. stock price of reinsurance companies and WTC 9/11 in 2001 (see Figure 1, with Munich Re and SCOR - European markets since Wall street has been closed after the catastrophe), oil price and Katrina in August 2005 ... etc.

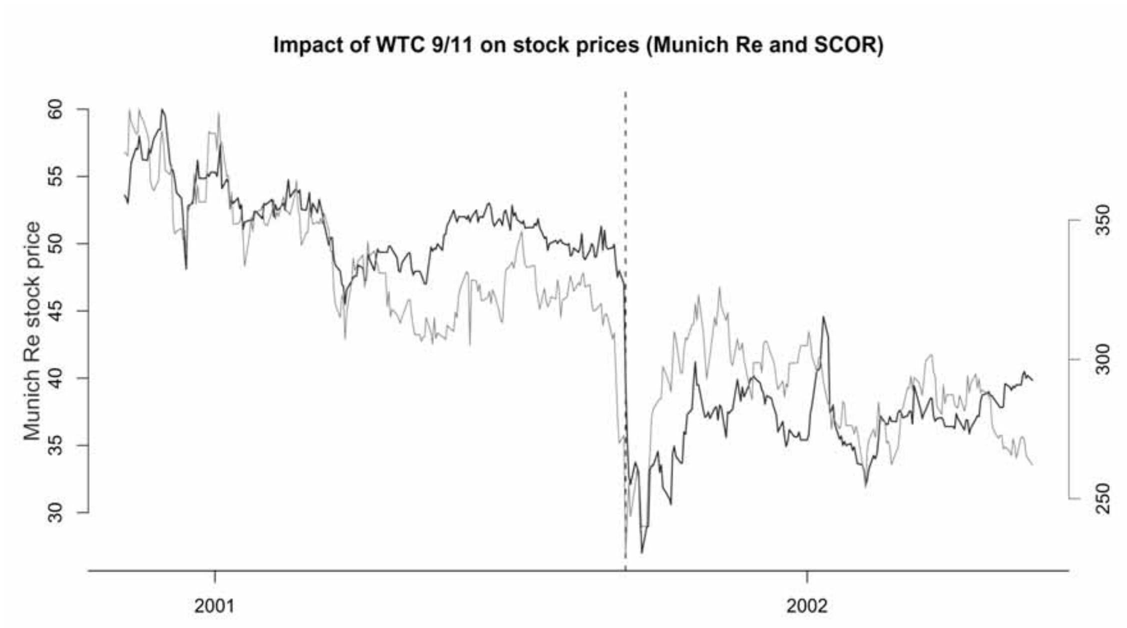

Figure 1: Catastrophe event and stock prices.

The following model and results are based on Quéma et al. (2007).

5.1. Description of the model

The occurrence process is a $(\mathcal{F}_t)$-adapted process denoted by $(N_t)_{t\geqslant 0}$. Under $\mathbb{P}$, assume that $(N_t)_{t\geqslant 0}$ is an homogeneous Poisson process, with parameter λ, i.e. with stationary and independent increments. Further, recall that $\mathbb{E}_{\mathbb{P}}(N_T) = \lambda T$ and $\text{Var}(N_T) = \lambda T$.

At time t the number of catastrophes that already occurred is N_t. Define the sequence of stopping times $(T_n)_{n\geqslant 0}$ corresponding the dates of occurrence of catastrophes, i.e.

$$T_0 = 0 \quad \text{and} \quad T_{n+1} = \inf\{t \mid t \geqslant T_n, N_t \neq N_{T_n}\}.$$

Let $(M_t)_{t\geqslant 0}$ be the compensated Poisson process of $(N_t)_{t\geqslant 0}$, i.e. $M_t = N_t - \lambda t$.

The i^{th} catastrophe has a loss modeled by a positive random variable $\mathcal{F}_{T_i}$-measurable denoted X_i. Variables $(X_i)_{i\geqslant 0}$ are supposed to be integrable, independent and identically distributed. Define $L_t = \sum_{i=1}^{N_t} X_i$ as the loss process, corresponding to the total amount of catastrophes occurred up to time t.

Assume that the financial market satisfies the no-arbitrage assumption, and consists in a risk free asset, and a risky asset, with price $(S_t)_{t\geqslant 0}$. Without loss of generality, the value of the risk free asset is assumed to be constant (hence it is chosen as a numeraire). The price of the risky asset is driven by the following diffusion process,

$$dS_t = S_{t^-}\left(\mu dt + \sigma dW_t + \xi dM_t\right) \text{ with } S_0 = 1$$

where $(W_t)_{t\geqslant 0}$ is a Brownian motion under $\mathbb{P}$, independent of the catastrophe occurrence process $(N_t)_{t\geqslant 0}$. Parameters μ and σ^2 are respectively the trend and the volatility of the risky asset, per time unit. Parameter ξ corresponds to the relative variation of the asset value when it jumps.

Note that the stochastic differential equation has the following explicit solution

$$S_t = \exp\left[\left(\mu - \frac{\sigma^2}{2} - \lambda\xi\right)t + \sigma W_t\right](1+\xi)^{N_t}.$$

5.2. Indifference utility

As in Davis (1997) or Schweizer (1997), assume that an investor has a utility function U, and initial endowment ω. The investor is trading both the risky asset and the risk free asset, forming a *dynamic* portfolio $\delta = (\delta_t)_{t\geqslant 0}$ whose value at time t is $\Pi_t = \Pi_0 + \int_0^t \delta_u dS_u = \Pi_0 + (\delta\cdot S)_t$ where $(\delta\cdot S)$ denotes the stochastic integral of δ with respect to S.

A strategy δ is admissible if there exists $M > 0$ such that $\mathbb{P}\Big(\forall t\in[0,T], (\delta\cdot S)_t \geqslant -M\Big) = 1$, and further if $\mathbb{E}_{\mathbb{P}}\left[\int_0^T \delta_t^2 S_{t^-}^2 dt\right] < +\infty$.

If X is a random payoff, the classical Expected Utility based premium is obtained by solving

$$u(\omega, X) = U(\omega - \pi) = \mathbb{E}_{\mathbb{P}}(U(\omega - X)).$$

Consider an investor selling an option with payoff X at time T,

- either he keeps the option, $u_{\delta^\star}(\omega, 0) = \sup_{\delta\in\mathcal{A}} E_{\mathbb{P}}\Big[U(\omega + (\delta\cdot S)_T)\Big]$,
- or he sells the option, $u_{\delta^\star}(\omega+\pi, X) = \sup_{\delta\in\mathcal{A}} \mathbb{E}_{\mathbb{P}}\Big[U(\omega + (\delta\cdot S)_T - X)\Big]$.

The price obtained by indifference utility is the minimum price such that the two quantities are equal, i.e.

$$\pi(\omega, X) = \inf \{\pi \in \mathbb{R} \mid u_{\delta^\star}(\omega + \pi, X) - u_{\delta^\star}(\omega, 0) \geqslant 0\}.$$

This price is the minimal amount such that it becomes interesting for the seller to sell the option: under this threshold, the seller has a higher utility keeping the option, and not selling it.

Based on optimal control results, Quéma et al. (2007) derived some analytical expression, that can be related to Merton (1976), in the case of exponential utility.

5.3. Following Merton's work

Assume that the asset has no jump, i.e. $dS_t = S_t(\mu dt + \sigma dW_t)$ (i.e. $\xi = 0$), and that we wish to price a derivative with payoff $\phi(S_T)$, then in the case of exponential utility $u_E(t,\pi) = U_E\left[\pi + \frac{\mu^2 x_0}{2\sigma^2}(T-t)\right]$.

In the case where the asset has jumps, i.e. $dS_t = S_{t^-}(\mu dt + \sigma dW_t + \xi dM_t)$ (i.e. $\xi \neq 0$), and that we wish to price a derivative with payoff $\phi(S_T)$, then $u_E(t,\pi) = U_E(\pi + (T-t)C)$ where $C(t)$ satisfies

$$\begin{cases} C(t) = \dfrac{\alpha x_0}{\xi} + (\alpha - \dfrac{\sigma^2}{\xi} - \xi\lambda)D - \dfrac{1}{2x_0}\sigma^2 D^2 \\ 0 = \xi\lambda - \alpha + \dfrac{\sigma^2}{x_0}D - \xi\lambda \exp\left[-\dfrac{\xi D}{x_0}\right] \end{cases},$$

with also a boundary condition, $C(T) = \phi(S_T)$, and where D is related to the optimal strategy, and is obtained also from the previous system.

Here, we wish to price a derivative with payoff $\phi(L_T)$, when the underlying asset has jumps. Then, assuming that the investor has an exponential utility, $U(x) = -\exp(-x/x_0)$, we find:

Theorem 5.1 *Let ϕ denote a $\mathcal{C}^2$ bounded function. If the utility function is exponential, the value function associated to the primal problem,*

$$u(t,\pi,s,l) = \max_{\delta\in\mathcal{A}} \mathbb{E}_{\mathbb{P}}\left[U\Big(\Pi_T - \phi(L_T)\Big) \mid \mathcal{F}_t\right]$$

does not depend on s and can be expressed as $u(t,\pi,l) = U\Big(\pi - C(t,l)\Big)$, where C is a function independent of π satisfying

$$\begin{cases} 0 &=& \xi\lambda - \mu + \dfrac{\sigma^2 s\delta^\star}{x_0} - \xi\lambda \exp\Big[-\dfrac{\xi s\delta^\star + C(t,l)}{x_0}\Big]\mathbb{E}_{\mathbb{P}}\left(e^{\frac{1}{x_0}C(t,l+X)}\right) \\ \dfrac{\partial C}{\partial t}(t,l) &=& \dfrac{\mu x_0}{\xi} + (\mu - \dfrac{\sigma^2}{\xi} - \xi\lambda)s\delta^\star - \dfrac{1}{2x_0}\sigma^2(s\delta^\star)^2 \\ C(T,l) &=& \phi(l) \end{cases}$$

where $\delta^\star$ denotes the optimal control.

Proof. Theorem 19 in Quéma et al. (2007). ■

5.4. Numerical issues and properties of optimal portfolios

From Theorem 5.1 we have to find the pair of functions (C, δ) as a solution to an integro-differential equation. Hopefully, using a simple discretization on a finite grid, it is possible to obtain a (stable) numerical approximation of C, and therefore of function C, and thus of the price of the derivative. Note further that two nice results have been derived in Quéma et al. (2007):

Lemma 5.2 *$C(t, \cdot)$ is increasing if and only if ϕ is increasing.*

Lemma 5.3 *If ϕ is increasing and $\mu > 0$, then the optimal amount of risky asset to be hold when hedging, is bounded from below by a strictly positive constant.*

For a numerical example, assume that the trend is null ($\mu = 0$), i.e. the amounts to be hold are uniquely determined by the hedging strategy. Prices are decreasing in x_0, and therefore, increasing with risk aversion (the higher x_0, the lower risk aversion), see Figure 2. When $x_0 \to 0$, risk aversion is infinite, and thus, whatever happens, the agent wants to hedge against any losses: the price tends to the super-replication price i.e. $\|\phi\|_\infty$, since if he holds the underlying, he might loose money.

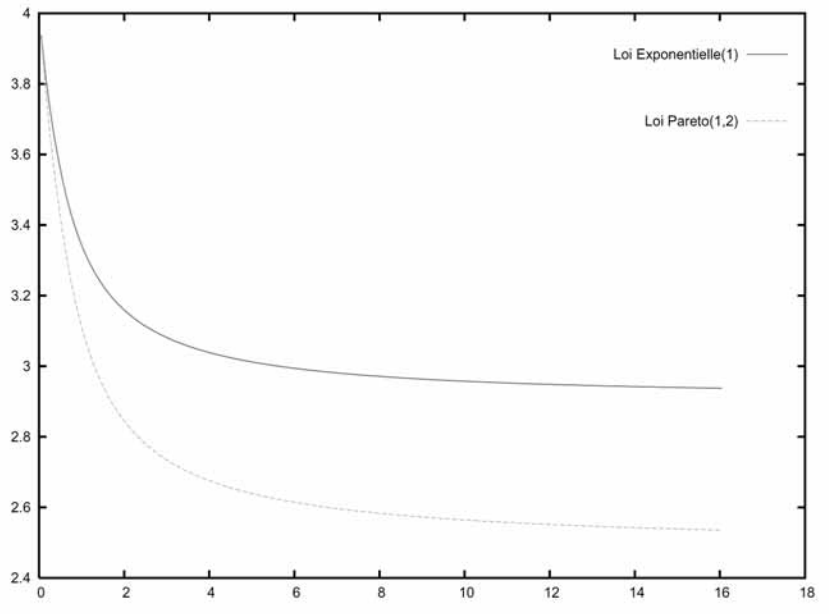

Figure 2: Price as a function of the risk aversion coefficient x_0 with $T = 1$, $\mu = 0$, $\sigma = 0.12$, $\lambda = 4$, $\xi = 0.05$ and $B = 4$

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HOW AMBIGUITY AFFECTS OPTIMAL REGULATION

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Abstract

In contrast to insurance companies, regulatory authorities or regulators typically have access to only limited information about the companies' value. It hence leads to some effect in the regulation design, which is however often overlooked in the literature. This paper characterizes the limited/imperfect information as Knightian uncertainty (ambiguity). In order to stress the analytical effects of ambiguity on the regulation decisions, we carry out an analysis in a simple default and a liquidation model setup, i.e., a standard immediate bankruptcy regulation. It is particularly noted that an ambiguity-averse regulator requires more "ambiguity equity". We show then that the standard immediate bankruptcy regulation may declare false liquidation under ambiguity. A new regulation rule is thus designed with an additional auditory process.

1. INTRODUCTION

This paper is concerned with the issue on the information transparency of regulators in the regulation rules design. In the standard insurance regulation literature, it is often supposed that regulators have full information on the insurer's asset value. The nullification of this assumption can be however justified by the fact that insurance companies have some firm-specific information and regulators are able to use only information external to the firm, for instance, some industry-wise measures. Indeed, the classical model rules out the situation where regulators are not sure of the likelihood of states of the world, i.e., the future asset evolution of each single insurance company. In other words, it adopts a strict assumption about regulators' belief: they are *perfectly certain* of the objective probability law of the state process and this belief is perfectly identical to the true probability law. However, regulators, as outsiders, may not have perfect confidence on the perceived probability measure for insurance companies' future asset value. With imprecise information, he may claim the possibility of other probability measures and has no idea of the true plausibility of those measures. This is exactly the ambiguity defined by Knight (1921). Through experiments and theoretical proofs, it has been long demonstrated that such uncertainty is not reducible to a single probability with known parameters and has a great effect on decisions. In the present paper, we incorporate ambiguity in a default and liquidation modelling setup. Our work hence focuses on the impact of ambiguity on the insurance regulations and on the design of proper regulation rules under ambiguity. To our knowledge, it is the first paper involving the ambiguity feature in an insurance modelling framework.

This paper first aims at showing the impact of the ambiguity on the regulation decisions. The analysis is carried out in a simplified framework as proposed in the literature (c.f. e.g. Grosen and Jørgensen (2002)) to model default and bankruptcy, i.e., a standard bankruptcy regulation which does not distinguish between default and bankruptcy. We mainly examine how regulators establish an optimal regulation (barrier) level as an intervention under ambiguity. The barrier level is chosen such that the default probability is restricted to some fixed limit, say, to have a default probability not higher than 5%. For instance, we show that risk-neutral but ambiguity-averse regulators determine the optimal barrier level by choosing the *worst* scenario in all possible probability measures. Ambiguity-averse regulators require regulated firms to carry more equity (to meet the default probability constraint), which is defined in this paper as "ambiguity equity". Both analytical and numerical results demonstrate that ambiguity obviously affects the default probability and hence the regulation parameter determination. In this sense, ambiguity issue should be taken into consideration when designing insurance regulation.

As a second contribution of this paper, we design a new default and liquidation model under ambiguity. The conventional default modelling framework has an obvious drawback under ambiguity. Bankruptcy may be declared according to regulators' knowledge of insurance company's asset value, even when the real firm value is still sufficiently high. Such situations cannot be avoided when default and liquidation are put into action simultaneously. Another regulation rule is hence required to well incorporate ambiguity. Instead of immediate liquidation, our default and liquidation formulation contains a regulatory auditing rules and hence prevents the mistaken liquidation resulted from ambiguity. In principle, firms can only be liquidated when the firm's asset value indeed falls below the barrier level. We then analyze the impact of debt ratio, the riskiness of the firm's asset and the auditing frequency on liquidation probability.

The model we develop in this paper follows two streams of literature. On the one hand, it is related to the insurance default and regulation literature. In the existing literature of default modelling, Briys and de Varenne (1994, 1997, 2001) first interpret the work of Merton (1974, 1989) in an insurance context, in which only default at maturity is possible. Grosen and Jørgensen (2000, 2002) extend the model by allowing possible default at any instant before maturity. Bernard et al. (2005, 2006) and Chen and Suchanecki (2007) further extend Grosen and Jørgensen by either incorporating stochastic interest rate or discussing more realistic bankruptcy procedures. In all the literature, the regulators act very passively in the sense of not taking any actions against the collapse of the insurance company when it runs into default. This is apparently counterfactual. In reality, regulators always try to help the insurance company not to go bust. Moreover, the regulation is still to be strongly strengthened, because the collapse of many insurance companies is closely related to insufficient regulations. For instance, the fall of First Executive Life Insurance Co. provides important lessons in regulation of life insurance companies[1]. To this end, Bernard and Chen (2008)

[1] C.f. Schulte (1991) provides an insider's view on the the fall of First Executive Life Insurance Co.

investigate how regulators establish regulatory rules to meet some regulatory objectives and finally to improve the benefits of policyholders.

On the other hand, our work is related to the work of decision making under ambiguity. The Ellsberg (1961) paradox first demonstrates that risk and uncertainty are (behaviorally) meaningfully distinct from each other. The ambiguity theory is further developed prominently by Schmeidler (1989) (the Expected Utility Theory) and Schmeidler and Gilboa (1989) (the Multiple Expected Utility theory). The present paper carries on the formulation of the second theory, using the continuous-time implementation by Chen and Epstein (2002). The incompleteness caused ambiguity is also applied by Nishimura and Ozaki (2007) in irreversible investment.

The remainder of this paper is organized in the following way: Section 2 gives the model framework and introduces Chen and Epstein (2002) model of ambiguity. The impact of ambiguity on the determination of the regulation rules is demonstrated in Section 3 by using the standard but simplified model of immediate bankruptcy regulation. Section 4 then designs a new audit regulatory rule under ambiguity. Numerical results are provided on how the riskiness of the firm, debt ratio and auditing frequency affect liquidation probability. Finally, Section 5 concludes the paper.

2. MODEL SETUP

Contract Design In this paper, we examine the regulation of an insurance company that offers only one type of policy. These contracts grant policyholders a participation clause in the surpluses of the company's value and a minimum interest rate guarantee at the maturity T, *when there is no premature default.* The guaranteed amount corresponds to an initial investment G_0 accumulated with a minimum continuously-compounded interest rate g.

$$G_T = G_0 e^{gT}. \tag{1}$$

As conventionally assumed in the literature, a premature default results in an immediate liquidation. A premature liquidation occurs when the firm's assets hit the barrier level (which is set by the external regulators) before the maturity date and is constructed as a first passage time problem. Generally, the critical barrier level B_t is set up by regulators. Since the firm's assets A_t should be high enough to meet the increasing guaranteed amount, it is not unreasonable to assume a time-increasing barrier:

$$B_t(\eta, \phi) = \eta G_0 e^{\phi t}, \quad \eta < \frac{A_0}{G_0} \tag{2}$$

for $t \in [0, T]$. The inequality $\eta < \frac{A_0}{G_0}$ ensures that the insurance company is solvent at the contract-issuing time. Here, both η and ϕ can be used by the regulator to control the strictness of the auditing rule. A high η or a high ϕ would lead to a high barrier level.

Ambiguity: Uncertainty in the Underlying Probability Measures According to Knight (1921), *risk* is the so-called "uncertainty" which can be reduced to a single probability; while the true *uncertainty* is not reducible. In the famous urn experiment[2], Ellsberg (1961) demonstrated that uncertainty, different from risk, has an impact on decisions: one prefers known to unknown or ambiguous probabilities. Besides in experimental settings, there are many obvious instances in our real life. For example, ambiguity is addressed to explain the equity premium puzzle whereby the representative agent model fails to fit historical averages of the equity premium and the risk-free interest rate. With uncertainty in the underlying probabilities, it is argued that part of the premium is indeed due to the greater ambiguity associated with the return to equity, which hence reduces the required degree of risk aversion. Meanwhile, Ellsberg's findings suggest that decision makers tend to be uncertainty or ambiguity averse. In the following, we formulate the ambiguity in such a way that the regulator is not sure that a given (i.e. predicted for the entire industry) probability measure is the true one, and that he assumes a set of probability measures instead and maximizes the minimum of expected payment to the insured based on each probability measure.

In case of ambiguity, the regulator is assumed to be able to observe only the insurer's initial firm value A_0 and terminal value A_T. The regulator's ambiguity about the firm's assets A_t, $t \in (0,T)$ is characterized by a set of priors $\mathcal{P}$ which are equivalent to the real world measure P. The set of the priors is determined by constructing a set of suitable densities which is further specified by a series of density generators Θ. Each density generator $(\theta_t) \in \Theta$ is assumed to be an $\mathbb{R}$-valued process for which the process (z_t^θ) is a P-martingale, i.e.

$$dz_t^\theta = -z_t^\theta \, \theta_t \, dW_t, \qquad z_0^\theta = 1,$$

where W_t is a Brownian motion under P and θ satisfies $E\left[\exp\left\{\frac{1}{2}\int_0^T |\theta|^2 ds\right\}\right] < \infty$, the Novikov condition. Solving this stochastic equation leads to

$$z_t^\theta = \exp\left\{-\frac{1}{2}\int_0^t |\theta_s|^2 ds - \int_0^t \theta_s \cdot dW_s\right\}, \quad 0 \le t \le T.$$

Given the set Θ of density generators, the corresponding set of priors is

$$\mathcal{P}^\Theta = \left\{Q^\theta : \theta \in \Theta \text{ and } Q^\theta \text{ is defined by } \frac{dQ^\theta}{dP} = z_T^\theta\right\},$$

which is mutually absolutely continuous with respect to P. Furthermore, all the candidate probability measures are naturally not widely from the industry-specific measure, but with a small deviation. In particular, the density generators $(\theta_t) \in \Theta$ are supposed to be restricted to the range $[-\kappa, \kappa]$. It is the so-called κ-ignorance (c.f. Chen and Epstein (2002)) and κ can be considered as the degree of Knightian uncertainty or ignorance.

Given any $\theta \in \Theta$, the regulator builds up a set of probability measures under which the firm's assets evolve as

$$dA_t^\theta = \mu \, A_t^\theta dt + \sigma A_t^\theta dW_t^\theta = (\mu + \sigma\theta_t) A_t^\theta dt + \sigma A_t^\theta dW_t. \tag{3}$$

[2]To roughly outline the Ellberg paradox, suppose that we have two urns. There are exactly 10 red and 10 blue balls in the first urn. The second one contains also altogether 20 balls either in red or in blue. However, there is no clear number of red or blue balls. The experiment is designed to check the willingness of persons who are offered a bet on drawing a red ball from the two urns. The result shows that a majority of persons choose to draw from the first urn rather than the second. The result remains when drawing a blue ball.

where μ and $\sigma > 0$ are instantaneous rate of return and volatility of the asset and the initial value $A_0^\theta = A_0$ takes the same value for all θ. Nevertheless, the real stochastic evolution of the firm's value is independent of the regulator's knowledge, following

$$dA_t = \mu A_t dt + \sigma A_t dW_t. \tag{4}$$

Clearly, those stochastic processes in the set $\mathcal{P}$ vary only in the drift term. The ambiguity in the probability measures gives rise to some different stochastic processes to specify or interpret. Moreover, their confidence over probability measures is supposed to be not (greatly) improved over time. Formally, it is not possible to learn the distribution of $\theta \in \Theta$, neither to reduce the set $\mathcal{P}$ over time.

3. OPTIMAL STANDARD IMMEDIATE BANKRUPTCY RULE UNDER AMBIGUITY

This section aims at investigating the impact of ambiguity on the regulation decisions analytically. To this end, a simple default and liquidation model setup is taken, which we call "standard immediate bankruptcy rule under ambiguity". Under this standard rule, the regulator declares liquidation of the firm immediately when the proxy A_t^θ breaches the predetermined default threshold, no matter how high the real firm's value A_t is. The premature default time or premature liquidation time is defined as

$$\tau^\theta := \inf\{t \le T \mid A_t^\theta \le B_t(\eta, \phi)\}. \tag{5}$$

Proposition 3.1 (Default Probability) *A "default-probability minimizing" regulator with Q^θ- beliefs, who is concerned with the default probability*

$$Q^\theta\left(\tau \le T\right), \tag{6}$$

under κ-ignorance, i.e., $(\theta_t) \in \Theta$ are restricted to the range $[-\kappa, \kappa]$, achieves the minimum default probability when the ambiguity parameter equals to the "lower-rim density generator" $-\kappa$

$$\min_{\theta\in\Theta} Q^\theta\left(\tau \le T\right) = Q^{-\kappa}\left(\tau \le T\right) = N\left(d_2^T(\hat{\mu}^*)\right) + \left(\frac{A_0}{\eta G_0}\right)^{\frac{-2\hat{\mu}^*}{\sigma^2}} N\left(d_1^T(\hat{\mu}^*)\right) \tag{7}$$

with $\hat{\mu}^* = \mu + \sigma\kappa - \phi - \frac{\sigma^2}{2}$. The maximum default probability is obtained when $\theta = \kappa$:

$$\max_{\theta\in\Theta} Q^\theta\left(\tau \le T\right) = Q^{\kappa}\left(\tau \le T\right) = N\left(d_2^T(\hat{\mu}^{**})\right) + \left(\frac{A_0}{\eta G_0}\right)^{\frac{-2\hat{\mu}^{**}}{\sigma^2}} N\left(d_1^T(\hat{\mu}^{**})\right) \tag{8}$$

with $\hat{\mu}^{**} = \mu - \sigma\kappa - \phi - \frac{\sigma^2}{2}$ and $d_{1/2}^T(\hat{\mu}) = \frac{1}{\sigma\sqrt{T}}[\ln(\frac{\eta G_0}{A_0}) \pm \hat{\mu}T]$.

Proof. See Chen and Xu (2008). ∎

The regulator will establish the regulation level, or more specifically the barrier level, in order to meet the solvency requirement. In this model, η and ϕ are the two regulation parameters. Trivially, the default probability decreases with both η and ϕ. Moreover, the barrier level becomes a constant equal to ηG_0, as ϕ approaches zero. Thus, we assume that the regulator aims at achieving a default probability constraint ε, which in turn helps us (numerically) determine the optimal regulation level η.

Proposition 3.2 (Ambiguity Attitude) *An ambiguity-averse regulator behaves according to the worst scenario rule. Namely, he chooses the optimal regulation level which makes the worst scenario — maximal default probability — below the default probability constraint*[3], *i.e.,*

$$\max_{\eta \text{ or } \phi} \max_{\theta \in \Theta} Q^\theta(\tau^\theta \leq T) \leq \varepsilon.$$

In contrast, an ambiguity-friendly regulator is taking $-\kappa$ degree of ignorance and chooses an optimal regulation level to make the most optimistic scenario — minimal default probability — binding, i.e.

$$\max_{\eta \text{ or } \phi} \min_{\theta \in \Theta} Q^\theta(\tau^\theta \leq T) = \varepsilon.$$

Proof. See Chen and Xu (2008). ■

ε	$\sigma = 0.10$			$\sigma = 0.15$		
	η^*	η^*_κ	$\eta^*_{-\kappa}$	η^*	η^*_κ	$\eta^*_{-\kappa}$
0.01	0.73818	0.68413	0.79078	0.47162	0.43378	0.51114
0.02	0.79008	0.73536	0.84197	0.52718	0.48629	0.56934
0.03	0.82347	0.76879	0.87437	0.56479	0.52211	0.60839
0.04	0.84867	0.79431	0.89853	0.59424	0.55032	0.63875
0.05	0.86915	0.81524	0.91799	0.61888	0.57405	0.66400
0.06	0.88653	0.83314	0.93436	0.64031	0.59478	0.68583
0.07	0.90169	0.84887	0.94855	0.65942	0.61335	0.70521
0.08	0.91519	0.86297	0.96110	0.67676	0.63027	0.72271
0.09	0.92738	0.87578	0.97238	0.69271	0.64589	0.73873
0.10	0.93853	0.88756	0.98263	0.70752	0.66045	0.75355

Table 1: Optimal η levels with parameters:
$A_0 = 100$, $G_0 = 80$, $T = 10$, $\mu = 0.06$, $g = 0.02$, $\kappa = 0.01/\sigma$, $\phi = 0.02$.

Table 1 and 2 provide some optimal η and ϕ values, respectively. η^* (ϕ^*), η^*_κ (ϕ^*_κ) and $\eta^*_{-\kappa}$ ($\phi^*_{-\kappa}$) denote the optimal η (ϕ) obtained for regulators with ambiguity-neutral, -averse and -friendly regulators. First, a positive relation between η and the default probability constraint ε is observed, and also that between η and ϕ. Besides, η (ϕ) declines with the volatility σ. A higher volatility leads to a higher default probability, as a compensation, either η or ϕ shall be set lower to meet the same default constraint ε. More importantly, regulators' attitude towards ambiguity delivers an obvious effect on the optimal η and hence the barrier level. Moving from the extreme case of $\theta = \kappa$ (ambiguity aversity) to $\theta = -\kappa$ (ambiguity-friendliness) results in an increase in the optimal η value by over 10% and a quite big difference in the barrier level between 6.4 and 8. Comparing η^*_κ and $\eta^*_{-\kappa}$ with η^*, if the regulator uses η^*_κ to determine the optimal barrier level, the real firm's

[3]Note that the default probability is observed (and calculated) by the regulator under measure Q instead of the real world probability measure P.

value falls below the barrier indeed with a smaller probability than ε. On the contrary, the real firm's value falls below the barrier with a larger probability than ε, while using $\eta^*_{-\kappa}$.

ε	$\sigma = 0.10$			$\sigma = 0.15$		
	ϕ^*	ϕ^*_κ	$\phi^*_{-\kappa}$	ϕ^*	ϕ^*_κ	$\phi^*_{-\kappa}$
0.01	0.06625	0.05625	0.07625	0.01278	0.00278	0.02278
0.02	0.07509	0.06509	0.08509	0.02660	0.01660	0.03660
0.03	0.08068	0.07068	0.09068	0.03532	0.02532	0.04532
0.04	0.08489	0.07489	0.09489	0.04185	0.03185	0.05185
0.05	0.08831	0.07831	0.09831	0.04715	0.03715	0.05715
0.06	0.09122	0.08122	0.10122	0.05165	0.04165	0.06165
0.07	0.09376	0.08376	0.10377	0.05559	0.04559	0.06559
0.08	0.09605	0.08605	0.10605	0.05911	0.04911	0.06911
0.09	0.09812	0.08812	0.10812	0.06231	0.05231	0.07231
0.10	0.10003	0.09003	0.11003	0.06525	0.05525	0.07525

Table 2: Optimal ϕ levels with parameters:
$A_0 = 100$, $G_0 = 80$, $T = 10$, $\mu = 0.06$, $\kappa = 0.01/\sigma$, $\eta = 0.5$.

Clearly, an ambiguity-averse regulator requires a much lower regulation level (and hence the barrier level) to achieve the same default probability. If an optimal regulation level η^* (optimal for an ambiguity-neutral regulator) is applied to an regulator who is indeed ambiguity-averse, a higher default probability (than the default probability constraint) results. In other words, an ambiguity-averse regulator will ask for some "ambiguity premium" (c.f. Chen and Epstein (2002) in the context of equity premium puzzle) to achieve the same default probability. This ambiguity premium could be e.g. a requirement to increase the investment of the equity holder, which decreases the default probability. We call this additional equity requirement "**ambiguity equity**". Table 3 illustrates the ambiguity equity as a function of the default probability constraint ε and of ignorance degree κ for three volatility levels. The higher the allowed default probability level ε, the more tolerant the regulator, the less ambiguity premium an ambiguity-averse regulator requires. Furthermore, the ambiguity equity increases in the degree of ignorance. That is, the more ambiguity averse the regulator is, the more ambiguity equity results. Please note that the ambiguity equity is indeed quite high overall. For instance, for a $\kappa = 0.7$ ($\sigma = 0.2$), the required equity shall be increased by 8.55075, which corresponds to an increase in equity by 42.75%. The ambiguity equity is then observed to rise with the volatility of the firm value.

Ambiguity is usually ignored in the literature and regulators considered in the most models are assumed to be ambiguity-neutral. The designed default probability ε is then not able to be achieved in reality where incomplete information or ambiguity is observed and especially when the regulator is ambiguity-averse. Ambiguity-averse regulators behave themselves more "conservatively" and admit the "worst-scenario" rule, which corresponds to the case $\theta = \kappa$ in this context where a maximum default probability results and the optimal regulation level turns out to be lower than the ambiguity-neutral case. As a result, by both theory and practice a special attention to ambiguity issues in the regulation design and optimal barrier level determination is called for.

ε	Ambiguity equity		
	$\sigma = 0.10$	$\sigma = 0.15$	$\sigma = 0.20$
0.01	5.85795	6.41898	6.65300
0.02	5.54101	6.20525	6.48772
0.03	5.31166	6.04629	6.36416
0.04	5.12418	5.9132	6.26013
0.05	4.96221	5.7957	6.16774
0.06	4.81775	5.68878	6.08317
0.07	4.68623	5.58958	6.00424
0.08	4.56474	5.49628	5.92958
0.09	4.45132	5.40768	5.85825
0.10	4.34457	5.3229	5.78959

κ	Ambiguity equity		
	$\sigma = 0.10$	$\sigma = 0.15$	$\sigma = 0.20$
$0.005/\sigma$	2.43446	2.90925	3.12005
$0.006/\sigma$	2.93338	3.48877	3.73554
$0.007/\sigma$	3.43583	4.06728	4.34808
$0.008/\sigma$	3.9416	4.64467	4.95766
$0.009/\sigma$	4.45046	5.22084	5.56422
$0.010/\sigma$	4.96221	5.7957	6.16774
$0.011/\sigma$	5.47662	6.36916	6.76818
$0.012/\sigma$	5.99347	6.94113	7.36552
$0.013/\sigma$	6.51256	7.51152	7.95972
$0.014/\sigma$	7.03365	8.08025	8.55075

Table 3: Ambiguity equity with parameters:
$A_0 = 100$, $G_0 = 80$, $T = 10$, $\mu = 0.06$, $g = 0.02$, $\kappa = 0.01/\sigma$, $\phi = 0.02$ and $\varepsilon = 0.05$ (right table).

4. REGULATORY AUDITING RULES UNDER AMBIGUITY

As easily observed, the regulation determination heavily depends on the regulator's estimation, A_t^θ, which is however not the true evolution of the insurance company's value A_t. Thus, under ambiguity, the standard immediate bankruptcy rule may declare a false bankruptcy.

Proposition 4.1 (False Bankruptcy Declaration) *An ambiguity-averse regulator (with κ-ignorance equal to κ) falsely declares liquidation, although the firm's asset does not hit the barrier, with the probability of*

$$\begin{aligned} P\left(\{\tau^{-\kappa} \leq T \ \ and \ \ \tau > T\}\right) &= N\left(\frac{D_0 - \hat{\mu}^* T}{\sigma\sqrt{T}}\right) + \left(\frac{A_0}{\eta G_0}\right)^{\frac{-2\hat{\mu}^*}{\sigma^2}} N\left(\frac{D_0 + \hat{\mu}^* T}{\sigma\sqrt{T}}\right) \\ &\quad -\left(N\left(\frac{D_0 - mT}{\sigma\sqrt{T}}\right) + \left(\frac{A_0}{\eta G_0}\right)^{\frac{-2m}{\sigma^2}} N\left(\frac{D_0 + mT}{\sigma\sqrt{T}}\right)\right), \end{aligned}$$

where $\tau := \inf\{t \leq T \mid A_t \leq B_t(\eta, \phi)\}$, $m = \mu - \phi - \frac{1}{2}\sigma^2$, $D_0 = \ln\frac{\eta G_0}{A_0}$, $\hat{\mu}^* = \mu + \sigma\kappa - \phi - \frac{\sigma^2}{2}$ *and P is the real world probability measure.*

Proof. See Chen and Xu (2008). ■

This probability can be analogously considered as Type-I error in statistics "the null hypothesis is correct but still be rejected". Table 4 analytically shows the probability of type I error in the standard immediate default and liquidation framework. The more risky the insurer's assets are, the more likely the Type I error occurs. Through this, the standard regulation rule turns out to be suboptimal with the time going on. This ex-post sub-optimality fact is due to the information lackage/ambiguity issue. Especially, we design in this section a more realistic regulation rule by introducing an auditing process, in order to mitigate the effect of ambiguity and the possible mistaken liquidation resulted from ambiguity.

σ	0.10	0.12	0.14	0.16	0.18	0.20
Probability	0.012	0.0282	0.0443	0.0566	0.0645	0.0684

Table 4: Probability of Type-I errors with
$A_0 = 100$, $G_0 = 80$, $T = 10$, $\mu = 0.06$, $g = 0.02$, $\kappa = 0.01/\sigma$, $\phi = 0.02$.

Auditory Regulation Design A regulatory auditing process against the insurer's firm value A_t is initiated whenever the monitored proxy hits the deterministic time-dependent barrier $B_t(\eta, \phi)$ over the monitoring times. In reality, monitoring and auditing cannot be carried out continuously due to high monitoring and/or auditing costs[4]. Therefore, in what follows, we suppose that the regulator monitors the proxy A_t^θ at a set of selected equidistant time points on $(0, T)$: $\mathcal{T} = \{t_1, t_2, \ldots, t_{N-1}\}$, $t_j = \frac{jT}{N}$, $j = 1, \ldots, N-1$, with $t_N = T$. Both proxy monitoring and regulatory auditing occur only at the time points containing in $\mathcal{T} \cup \{0, T\}$.

Suppose that it holds that $A_{\bar{t}}^\theta \leq B_{\bar{t}}(\eta, \phi)$ for the first time at some particular $\bar{t} \in \mathcal{T}$. Then, the auditing process is immediately in effect and there are two possible auditing outcomes at $\bar{t}$:

1. $A_{\bar{t}} \leq B_{\bar{t}}(\eta, \phi)$,
2. and $A_{\bar{t}} > B_{\bar{t}}(\eta, \phi)$.

We assume that at time $\bar{t}$ the regulator resets $A_{\bar{t}}^\theta$ to the insurer's true firm value immediately, i.e. $A_{\bar{t}+}^\theta = A_{\bar{t}}$. When case 1 occurs, the insurer is liquidated immediately as the insurer's true firm value has hit some "dangerous" level from the regulator's viewpoint. An immediate liquidation is therefore called for to protect the policyholder from further loss. Under scenario 2, the liquidation is not to be carried out since the insurer's prevailing true value is above the auditing level set by the regulator. Upon updating the insurer's proxy, the auditing process is stopped and the regulator resumes its usual monitoring process against the proxy. Note that in this case the auditing process may be initiated again at a later time depending on the future evolution of the proxy. Though the regulator has full access to all historical data of A_t for $t < \bar{t}$ at time $\bar{t}$, we have assumed that the regulator has to "forgive" and "forget" even if the regulator finds $A_t < B_t(\eta, \phi)$ for some $t < \bar{t}$ in situation 2. This assumption is not completely unreasonable since only the prevailing value is relevant to the insurer's insolvency at the current moment, but not its past values. Let τ_L denote the liquidation time of the insurer. The specification above allows us to write

$$\tau_L = \inf \left\{ t \in \mathcal{T} \cup \{0, T\} \mid A_t^\theta \leq B_t(\eta, \phi), A_t \leq B_t(\eta, \phi) \right\}.$$

Under this setup, the auditing triggering event is not equivalent to the liquidation event.

Numerical Results Monte Carlo simulation is applied to approximate the newly designed auditory regulation rule. Throughout the simulation, a sample size of 100,000 is used.

The analysis is firstly carried out to investigate the impacts of the volatility σ under different debt ratio (D/E) and Table 5 provides the corresponding values. With $A_0 = 100$, assuming a D/E of 4 and 1 is equivalent to setting G_0 to 80 and 50, respectively. From the table, it is clear that the

[4]Cost of enforcement includes both the necessary public funding (e.g. salaries of bureaucrats and judges, paperwork, investigations, etc.) and the compliance costs borne by audit firms.

D/E has a great impact on the liquidation probability. For example, for $\sigma = 0.16$ and $\kappa = 0.01/\sigma$, default happens to the firm with a probability of 14.09% when $D/E = 4$ (under the regulation of an ambiguity-friendly regulator), whereas the probability reduces to 1.39% when $D/E = 1$. This suggests that the insurer has a high probability to be liquidated if its initial equity contribution is too low compared to its initial debt. It interprets exactly the ambiguity equity introduced in Section 3.

	D/E	σ					
		0.10	0.12	0.14	0.16	0.18	0.20
LP (Amb.-friendly)	4	0.0171	0.0472	0.0864	0.1409	0.1848	0.237
($\kappa = 0.01/\sigma$)	1	0.0001	0.001	0.0063	0.0139	0.0304	0.0487
LP (Amb.-averse)	4	0.0308	0.0698	0.1204	0.1724	0.2388	0.2844
($\kappa = 0.01/\sigma$)	1	0.0005	0.0021	0.008	0.0203	0.0401	0.0719
LP (Amb.-friendly)	4	0.0082	0.0323	0.0688	0.1134	0.1668	0.2084
($\kappa = 0.02/\sigma$)	1	0.0001	0.0011	0.0037	0.0098	0.0238	0.0424
LP (Amb.-averse)	4	0.0346	0.0804	0.1304	0.1881	0.2494	0.3003
($\kappa = 0.02/\sigma$)	1	0.0002	0.0039	0.0112	0.0244	0.0456	0.0665
LP (Amb.-friendly)	4	0.0006	0.0079	0.0239	0.0516	0.0825	0.133
($\kappa = 0.05/\sigma$)	1	0.0001	0.0002	0.0004	0.0032	0.0078	0.0184
LP (Amb.-averse)	4	0.0400	0.0988	0.1492	0.2194	0.2887	0.3397
($\kappa = 0.05/\sigma$)	1	0.0003	0.0033	0.0112	0.0259	0.0514	0.0853

Table 5: Liquidation probabilities (LP) for different σ values with parameters: $A_0 = 100$, $\eta = 0.8$, $\mu = 0.05$, $r = 0.03$, $T = 10$, $N = 10$, $g = 0.02$, $\phi = 0.03$.

As expected, the default probabilities are increasing in σ. For $\sigma = 0.1$, quite low liquidation probability results. However, a lower liquidation probability does not necessarily mean that the policyholder is better protected. For example, the regulator can always achieve a lower liquidation probability by reducing the magnitude of η or ϕ. By doing so, however, the amount of payment received by the policyholder in the event of liquidation is expected to be lower. Hence, the policyholder is less protected from this viewpoint. In practice, the bankruptcy of a large-sized insurer can leave many households in financial distress which imply a large amount of settlement, administration, and legal costs born by the regulator. Hence, it is nevertheless of great importance for the regulator to ensure that the insurer has a reasonably low shortfall probability. If one takes the insurer' standpoint, then a lower shortfall probability, of course, is always advantageous.

the κ-value determines the attitude of the regulator towards ambiguity. According to the analysis of Section 3, ambiguity-averse regulators act following the worst-scenario rule, whereas ambiguity-friendly regulators act extremely optimistically. The more ambiguity-averse the regulator is, the higher the default probability turns out to be. In contrast, the more ambiguity-friendly the regulator is, the smaller the liquidation probability results. An ambiguity-averse regulator uses an underestimated proxy for monitoring and auditing, which leads to more frequent auditing processes. Consequently, it is more likely that the company is found to be bankrupt. Certainly, a reversed reasoning shall hold for the ambiguity-friendly regulators. An ambiguity-neutral regulator behaves moderately and causes a shortfall probability in between.

	D/E	N		
		10	20	40
LP (Amb.-averse)	4	0.1724	0.2177	0.2509
($\kappa = 0.01/\sigma$)	1	0.0203	0.0245	0.0250
LP (Amb.-friendly)	4	0.1409	0.1717	0.2023
($\kappa = 0.01/\sigma$)	1	0.0139	0.0154	0.0196
LP (Amb.-averse)	4	0.1881	0.2349	0.2671
($\kappa = 0.02/\sigma$)	1	0.009	0.009	0.0140
LP (Amb.-friendly)	4	0.1134	0.1395	0.1636
($\kappa = 0.02/\sigma$)	1	0.0244	0.0273	0.0294

Table 6: Liquidation probabilities (LP) for different N values with parameters: $A_0 = 100$, $\eta = 0.8$, $\mu = 0.05$, $r = 0.03$, $\sigma = 0.16$, $T = 10$, $g = 0.02$, $\phi = 0.03$.

5. CONCLUSION

This paper highlights the crucial effect of ambiguity, i.e., the imperfect information the regulator holds about the insurance company's future asset's evolution, on insurance regulator's optimal regulation decisions and therefore stresses the transparency of the insurance undertaking. We calculate ambiguity equity for an ambiguity-averse regulator. Furthermore, the paper provides a possibility of how to incorporate ambiguity into a realistic default and liquidation (regulation) framework.

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OPTIMAL TRADING STRATEGIES IN PRESENCE OF MARKET IMPACTS

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Abstract

We consider an arbitrageur who takes advantage of market misvaluations. The market misvaluation is assumed to be mean reverting around zero and to be affected by the arbitrageur's actions. We propose a simple realistic model and explicitly solve the problem of finding optimal trading schemes.

1. INTRODUCTION

A major type of arbitrage involves taking long and short positions in different assets, the so-called long/short strategies. These strategies often take the form of a spread trade, where, for example, the arbitrageur may go long a particular stock which he or she believes is undervalued with respect to some market factors, and short a certain number of these factors in a way that zeros out the exposure to global market moves.

First, some type of analysis is applied to identify stocks that are undervalued and market factors. To fix ideas, one can think of individual high tech stocks, in which case, factors would be Microsoft shares and/or the Nasdaq index. Second, the investor enters into a portfolio that exploits these perceived misvaluations by going long the stock and short the factors in a way that minimizes the risk of the portfolio with respect to global market moves. Finally, the portfolio is held until the trade converges and the relative value of the stock comes back into line with that of the factors.

A very detailed presentation of these strategies in fixed income can be found in Duarte et al. (2007).

Once the anomaly has been found, a major issue of the hedge funds industry is to implement that strategy. Indeed, due to liquidity issues, especially when trading the undervalued stock, these trades often have a market impact that makes misvaluations disappear as soon as they are exploited. Arbitrageurs must limit the size and speed of the trade execution so that they take advantage of the misvaluation. Market practioners also face transaction costs, which is another, although different, market imperfection.

The present paper first presents a simple model for such arbitrage opportunities and for two market imperfections: transaction costs and market impact. Then, we are going to compute the optimal arbitrage strategy. The model for the misvaluations is based on the Ornstein-Uhlenbeck process since in the mind of the arbitrageur, the misvaluations tend to mean revert around zero.

1.1. Modelling the market misvaluation

To fix ideas, let us consider a stock S that is misvalued relative to a certain number of traded factors F^i. Let us imagine an arbitrageur who believes in a linear regression model for explaining the return on the stock by means of the returns on the factors. This can be written as

$$\frac{dS_t}{S_t} = \sum_{i=1}^{n} \beta^i \frac{dF_t^i}{F_t^i} + \frac{d\varepsilon_t}{S_t},$$

where β^i are regression coefficients and ε_t is the error term normalized by the stock price. This error term is also the value of a self-financing portfolio based on going long one share and short $\beta^i S_t / F_t^i$ of the ith factor at each time t:

$$d\varepsilon_t = dS_t - \sum_{i=1}^{n} \frac{\beta^i S_t}{F_t^i} dF_t^i.$$

We will assume that this error term has the Ornstein-Uhlenbeck dynamics:

$$d\varepsilon_t = a(b - \varepsilon_t)dt + \sigma dW_t \tag{1}$$

where W is a Brownian motion, σ is the volatility, a is the mean reversion speed and b is the mean reversion level. The case $b = 0$ is most interesting for the situation at hand and we will henceforth assume $b = 0$.

Suppose that at time $t = 0$, our arbitrageur finds that

$$\varepsilon_0 = S_0 - \sum_{i=1}^{n} \frac{\beta^i S_0}{F_0^i} F_0^i = \left(1 - \sum_{i=1}^{n} \beta^i\right) S_0 < 0$$

and decides to implement a self-financing strategy by holding φ_t of the above portfolio at each time t. Assuming a constant risk free rate $r \geq 0$, the value V_t of the strategy satisfies

$$dV_t = rV_t dt + \varphi_t(d\varepsilon_t - r\varepsilon_t dt)$$

since ε_t is itself the value of a self-financing strategy. We did not take dividends into account but these are easily incorporated if necessary.

Due to market impact and transaction costs, the arbitrageur faces a trade off: either he/she acts too rapidly and his/her impact and transaction costs make the strategy unprofitable, or he/she acts too slowly and the opportunity may disappear before he/she can make a profit.

1.2. Modelling the market impacts

1.2.1. Transaction costs

We choose to model the transaction costs in the following way. We assume that they are due to the speed at which the trade is executed, more precisely they induce a loss of

$$\frac{\lambda}{2}\dot{\varphi}_t^2$$

dollars per unit of time, where $\dot{\varphi}_t$ represents the time derivative of φ_t. The parameter λ is positive and depends on the particular market.

To understand the model's rational, let us look at a strategy that is discretely rebalanced. Trading at the bid/ask prices instead of the mid-price introduces an overhead cost of $\lambda' |\Delta\varphi_t|$ dollars where λ' is half the bid/ask spread and $\Delta\varphi_t$ is the change in stock holdings between t and $t + \Delta t$. In the continuous time limit, the overhead cost becomes a rate of $\lambda' |\dot{\varphi}_t|$ dollars per unit time. Our model therefore assumes that the bid/ask spread will depend on the execution's speed. It is an attempt to better take into account the market microstructure. Indeed, only a finite number of shares are available at the bid and ask prices. By selling or purchasing more than that number, the actual overhead cost will be higher. That effect is called "temporary market impact" in Almgren and Chriss (1999, 2000). One could easily propose more elaborate models. We aim here at transparency and simplicity.

1.2.2. Market impact on the stock dynamics

In addition to transaction costs, our strategy may influence the price evolutions. To this end, we make the dynamics of ε dependent on $\dot{\varphi}$, the execution's speed. The model then introduces a change in the long term mean reversion level proportional to $\dot{\varphi}_t$.

$$d\varepsilon_t = (-a\varepsilon_t + \mu\dot{\varphi}_t)dt + \sigma dW_t, \tag{2}$$

with μ a positive parameter depending on the particular market. It says that if we buy rapidly, $\dot{\varphi}_t > 0$, the mean reversion level will be positive, and the market imperfection will tend to return faster to zero. That second effect is often called "permanent market impact", see for instance Almgren and Chriss (1999, 2000).

Here again, one could easily propose more sophisticated models, where, for instance the trading scheme affects the mean reversion speed as well as the mean reversion level.

1.3. Finding the optimal trading strategies

Let us fix a time horizon T. Due to the market imperfections, our arbitrageur would like to maximize his discounted terminal wealth. He faces a stochastic optimization problem of the type

$$\sup_{\{\varphi_t\}} \mathbb{E}\left\{e^{-rT}V_T\right\}.$$

We did not include a penalty due to the risk of the position, but as we will see later, it can easily be incorporated. In the present analysis, we assume that the trader only tries to maximize his or her expected terminal wealth.

The above problem can be rewritten as

$$\sup_{\{\varphi_t\}} \mathbb{E}\left\{\int_0^T e^{-rt}\left[\varphi_t(d\varepsilon_t - r\varepsilon_t dt) - \frac{\lambda}{2}\dot{\varphi}_t^2 dt\right]\right\}, \tag{3}$$

where ε_t has dynamics given by (2). The supremum is taken over admissible strategies that will be defined for each problem.

Very often, we will assume that $\varphi_0 = 0$, which simply says that the arbitrageur has no position in the different assets at the beginning of time, but our results can easily be changed to handle the case $\varphi_0 \neq 0$. On the other hand, we would also like to be able to add the constraint $\varphi_T = 0$, which imposes that we exit the trade at the horizon T. We will treat the cases with and without that additional constraint separately.

2. SOLUTION TO THE OPTIMIZATION PROBLEM

Let W be a standard Brownian motion on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Its filtration is denoted by $(\mathcal{F}_t)_{t\geq 0}$.

2.1. The Bellman equation

In the full observation case, the problem faced by our arbitrageur can be written as in equation (3). We now need to carefully define the set of admissible strategies. These are strategies φ_t which are $(\mathcal{F}_t)$ adapted and almost surely absolutely continuous with derivative denoted by $\dot{\varphi}_t$. Moreover they satisfy the following integrability conditions

$$\mathbb{E}\left\{\int_0^T \varphi_t^2 + \dot{\varphi}_t^2 dt\right\} < \infty$$

for both Itô and Lebesgue integrals to be well-defined. Due to (2), for such strategies the problem can be rewritten as

$$\sup_{\{\varphi_t\}} \mathbb{E}\left\{\int_0^T e^{-rt}\left[-(a+r)\varepsilon_t\varphi_t + \mu\varphi_t\dot{\varphi}_t - \frac{\lambda}{2}\dot{\varphi}_t^2\right] dt\right\}.$$

In order to solve this problem, we introduce the following bivariate controlled diffusion X:

$$X_t = \begin{bmatrix} \varepsilon_t \\ \varphi_t \end{bmatrix}.$$

It has the following dynamics:

$$dX_t = \begin{bmatrix} -a\varepsilon_t + \mu\alpha_t \\ \alpha_t \end{bmatrix} dt + \begin{bmatrix} \sigma \\ 0 \end{bmatrix} dW_t,$$

where $\alpha = \dot{\varphi}$ is the control of the problem. Using matrix notation, the model can be expressed as

$$dX_t = (MX_t + \alpha_t N)dt + \Sigma dW_t$$

with

$$M = \begin{bmatrix} -a & 0 \\ 0 & 0 \end{bmatrix} \quad N = \begin{bmatrix} \mu \\ 1 \end{bmatrix} \quad \Sigma = \begin{bmatrix} \sigma \\ 0 \end{bmatrix}.$$

Moreover, the set of admissible controls $\mathcal{A}$ is characterized by

$$\mathcal{A} = \left\{ \alpha \mid \alpha \text{ is } (\mathcal{F}_t) \text{ adapted and } \mathbb{E}\left\{ \int_0^T \left(\int_0^t \alpha_s ds \right)^2 + \alpha_t^2 dt \right\} < \infty \right\}.$$

The problem is now cast in the following familiar form (see, for instance, Bensoussan (1992) or Øksendal (2003)):

$$\sup_{\alpha \in \mathcal{A}} \mathbb{E}\left\{ \int_0^T e^{-rt} \left[-\frac{1}{2} X_t' A X_t + \alpha_t B' X_t - \frac{\lambda}{2} \alpha_t^2 \right] dt \right\}, \tag{4}$$

with

$$A = \begin{bmatrix} 0 & a+r \\ a+r & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ \mu \end{bmatrix}.$$

Starting from this formulation, we introduce the indirect value function v:

$$v(t, X) = \sup_{\alpha \in \mathcal{A}} \mathbb{E}\left\{ \int_t^T e^{-ru} \left[-\frac{1}{2} X_u' A X_u + \alpha_u B' X_u - \frac{\lambda}{2} \alpha_u^2 \right] du \middle| X_t = X \right\};$$

the Bellman equation then reads

$$\sup_{\alpha} \left[\frac{\partial v}{\partial t} + (MX + \alpha N)' \nabla v + \frac{1}{2} \Sigma' \nabla^2 v \Sigma - rv - \frac{1}{2} X' A X + \alpha B' X - \frac{\lambda}{2} \alpha^2 \right] = 0 \tag{5}$$

with ∇v and $\nabla^2 v$ denoting the gradient and the Hessian with respect to the variable X. The terminal condition is $v(T, X) = 0$.

2.2. Explicit solution

The expression between brackets in equation (5) is quadratic in α and the supremum can be computed explicitly. This leads to the following nonlinear partial differential equation (PDE) for the value function v.

$$\frac{\partial v}{\partial t} + X' M \nabla v + \frac{1}{2} \Sigma' \nabla^2 v \Sigma - rv - \frac{1}{2} X' A X + \frac{1}{2\lambda} (B' X + N' \nabla v)^2 = 0,$$

with terminal condition $v(T, X) = 0$. The optimal feedback policy is

$$\alpha^* = \frac{1}{\lambda} (N' \nabla v + B' X).$$

We are looking for a solution of the form

$$v(t,X) = \frac{1}{2}X'\Omega_t X + \beta_t,$$

where Ω_t is a deterministic function of time taking values in 2×2 symmetric matrices and β is a scalar deterministic function. It can be easily seen that Ω must solve a matrix Riccati equation

$$\dot{\Omega}_t - A + \frac{BB'}{\lambda} + \Omega_t\left(M - \frac{r}{2}I + \frac{NB'}{\lambda}\right) + \left(M - \frac{r}{2}I + \frac{BN'}{\lambda}\right)\Omega_t + \Omega_t\frac{NN'}{\lambda}\Omega_t = 0$$

with terminal condition $\Omega_T = 0$, and with I denoting the identity matrix. Furthermore, β solves the linear ordinary differential equation

$$\dot{\beta}_t - r\beta_t + \frac{1}{2}\Sigma'\Omega_t\Sigma = 0$$

with terminal condition $\beta_T = 0$. Once the Riccati equation has been solved, β is given by

$$\beta_t = \frac{1}{2}\int_t^T e^{r(t-u)}\Sigma'\Omega_u\Sigma du.$$

Proposition 2.1 *The optimal strategy for problem* (4) *is given by*

$$\alpha_t^* = \frac{1}{\lambda}(N'\Omega_t + B')X_t,$$

where Ω is the solution to the matrix Riccati equation

$$\dot{\Omega}_t - A + \frac{BB'}{\lambda} + \Omega_t\left(M - \frac{r}{2}I + \frac{NB'}{\lambda}\right) + \left(M - \frac{r}{2}I + \frac{BN'}{\lambda}\right)\Omega_t + \Omega_t\frac{NN'}{\lambda}\Omega_t = 0$$

with terminal condition $\Omega_T = 0$. The value function is given by

$$v(t,\varepsilon_t,\varphi_t) = \frac{1}{2}X_t'\Omega_t X_t + \frac{1}{2}\int_t^T e^{r(t-u)}\Sigma'\Omega_u\Sigma du.$$

Proof. Since we found a solution to the Bellman equation for the problem, the proof follows from the classical verification theorem. ■

Introducing a finite horizon T may not always be relevant for certain applications. In such cases, it is interesting to know that as T goes to infinity, the value function v has a limit given in terms of the solution Ω to the algebraic Riccati equation

$$-A + \frac{BB'}{\lambda} + \Omega\left(M - \frac{r}{2}I + \frac{NB'}{\lambda}\right) + \left(M - \frac{r}{2}I + \frac{BN'}{\lambda}\right)\Omega + \Omega\frac{NN'}{\lambda}\Omega = 0.$$

Indeed,

$$\lim_{T\to+\infty} v(0,\varepsilon,\varphi) = \frac{1}{2}X'\Omega X + \frac{1}{2r}\Sigma'\Omega\Sigma.$$

2.3. Numerical example

The present section consists of a numerical example that will help us to understand the role of the market impact parameter μ on the optimal strategy.

We shall use the following set of parameters: the initial misvaluation $\varepsilon_0 = -1$, the mean reversion speed $a = 2$, the volatility $\sigma = 100\%$ per annum and the horizon is set to $T = 3$ years. We start with $\varphi_0 = 0$ in stock, and we take the transaction cost parameter $\lambda = 1$.

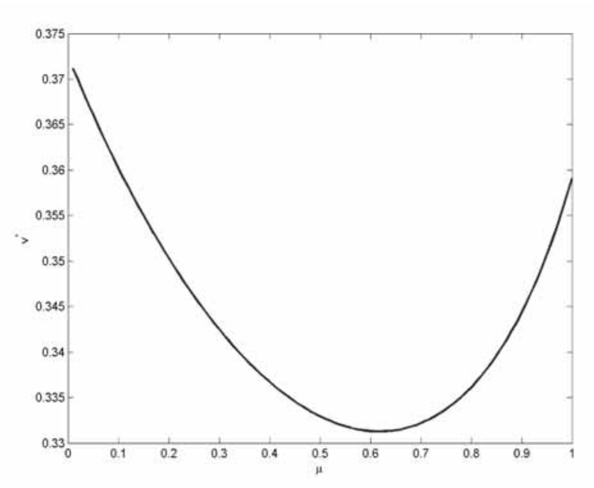

Figure 1: How the value function changes with respect to the market impact parameter μ.

We first study the impact on the value function of the market impact parameter μ. As shown in Figure 1, for values of μ less than 0.6, the value function decreases as a function of μ. This is expected since market impacts lower our ability to take advantage of the misvaluation. It is therefore quite surprising at first to see that for values of μ larger than 0.6, the value function is actually increasing in μ. However, this can be explained as follows: when μ is large enough, the arbitrageur has a large impact on the stock price movements, and he or she may decide to push the price further down. This is illustrated in Figures 2 and 3.

Second, we consider for a given outcome ω, two different values for the market impact parameter $\mu = 0.2$ or $\mu = 1$, and two different values for the initial number of shares $\varphi_0 = 0$ or $\varphi_0 = -3$. For a small value of the market impact parameter, $\mu = 0.2$, changing φ_0 has little qualitative effect on the optimal strategy in the first 2 years. This can be seen by comparing the plots in Figure 2 where we plotted against time, the misvaluation path (in blue), the misvaluation path when there is no market impact (in green), and with values on the right, the optimal number of shares held in the portfolio φ_t^*.

On the other hand, if we had started with a large market impact parameter $\mu = 1$, changing $\varphi_0 = 0$ to $\varphi_0 = -3$ changes the qualitative shape of the optimal strategy. With a large short initial position, it becomes optimal to further short the stock. This is seen in Figure 3. For reasonable practical situations, the parameter μ should therefore not be too large.

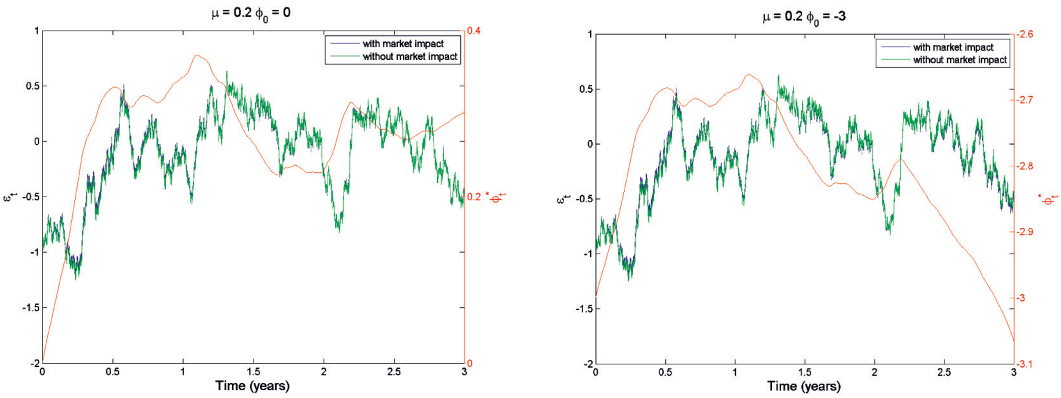

Figure 2: $\mu = 0.2$ and $\varphi_0 = 0$ (left) $\varphi_0 = -3$ (right). Note the different scales for φ^* (on the right of the plot) and for ε (on the left).

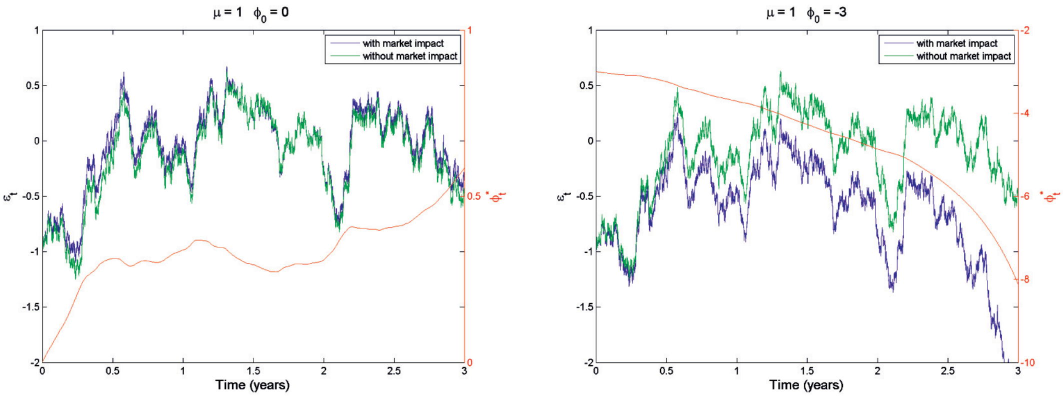

Figure 3: $\mu = 1$ and $\varphi_0 = 0$ (left) $\varphi_0 = -3$ (right). Note the different scales for φ^* (on the right of the plot) and for ε (on the left).

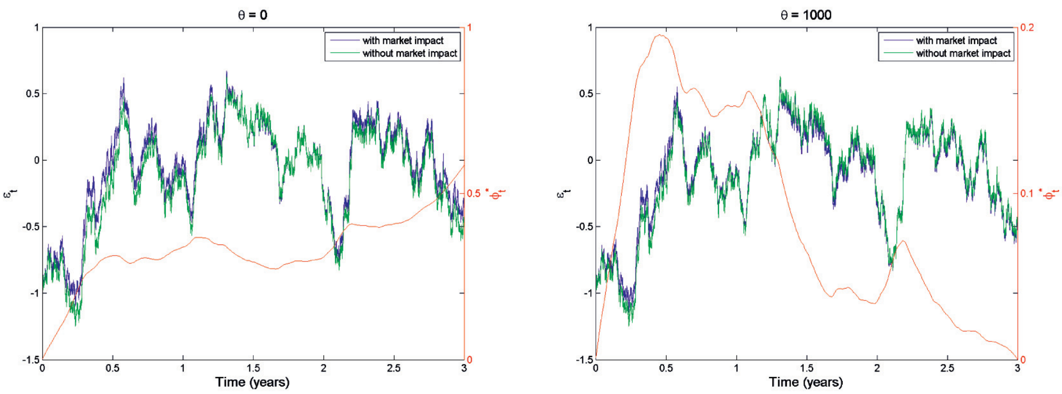

Figure 4: Optimal strategy with no constraint on φ_T (left) and with the constraint $\varphi_T = 0$ (right). Note the different scales for φ^* (on the right of the plot) and for ε (on the left).

2.4. Adding the constraint $\varphi_T = 0$

As shown in the numerical example above, it might be desirable to make sure that the positions of the arbitrageur have been closed by time T. This would rule out strategies where the arbitrageur with a short position pushes the stock price down as to increase his or her marked-to-market value. The classical way to add the constraint $\varphi_T = 0$ consists of introducing a Lagrange multiplier θ in the objective function. Keeping the same notations as before, we reformulate the stochastic control problem as

$$\sup_{\{\varphi_t\}} \mathbb{E}\left\{\int_0^T e^{-rt}\left[-(a+r)\varepsilon_t\varphi_t + \mu\varphi_t\dot{\varphi}_t - \frac{\lambda}{2}\dot{\varphi}_t^2\right]dt - \theta\varphi_T^2\right\}.$$

It can also be cast into the familiar form

$$\sup_{\alpha\in\mathcal{A}} \mathbb{E}\left\{\int_0^T e^{-rt}\left[-\frac{1}{2}X_t'AX_t + \alpha_t B'X_t - \frac{\lambda}{2}\alpha_t^2\right]dt - X_T'\Theta X_T\right\} \tag{6}$$

with

$$\Theta = \begin{bmatrix} 0 & 0 \\ 0 & \theta \end{bmatrix}.$$

Then proposition 2.1 easily extends to

Proposition 2.2 *The optimal strategy for problem* (6) *is given by*

$$\alpha_t^* = \frac{1}{\lambda}(N'\Omega_t + B')X_t,$$

where Ω *is the solution to the matrix Riccati equation*

$$\dot{\Omega}_t - A + \frac{BB'}{\lambda} + \Omega_t\left(M - \frac{r}{2}I + \frac{NB'}{\lambda}\right) + \left(M - \frac{r}{2}I + \frac{BN'}{\lambda}\right)\Omega_t + \Omega_t\frac{NN'}{\lambda}\Omega_t = 0$$

with terminal condition $\Omega_T = \Theta$. *The value function is given by*

$$v(t, \varepsilon_t, \varphi_t) = \frac{1}{2}X_t'\Omega_t X_t + \frac{1}{2}\int_t^T e^{r(t-u)}\Sigma'\Omega_u\Sigma du.$$

For θ going to ∞, the solution of (6) converges to the solution of (4) with the additional constraint $\varphi_T = 0$ a.s. This is illustrated in Figure 4, where the parameters are the same as in the previous section, with $\mu = 1$ and $\varphi_0 = 0$.

It is also interesting to see how the dependence of the value function on μ changes when we have the additional constraint. Not surprisingly, the value function is lower than in Figure 1 and it is now decreasing in μ (see Figure 5).

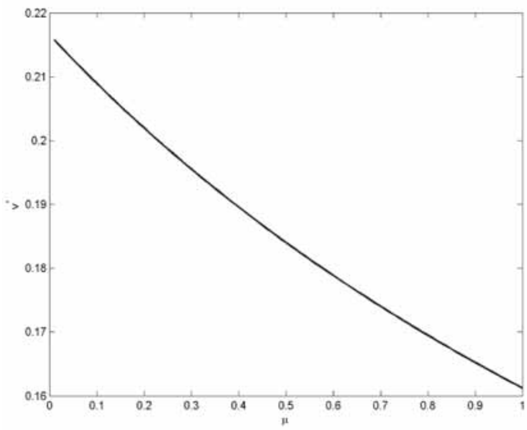

Figure 5: How the value function changes with respect to the market impact parameter μ with the additional constraint $\varphi_T = 0$.

2.5. Extensions

The objective of the arbitrageur in equation (4) only takes into account the expectation of the terminal value of the strategy. Like any other agent, the arbitrageur is risk averse and he/she may want to incorporate his/her attitude towards risk in his/her objective function. A simple way of achieving this in our model is to introduce a penalty term of the form

$$-\gamma \int_0^T \sigma^2 \varphi_t^2 dt$$

where γ is a risk aversion parameter. The integral is the quadratic variation of the wealth process V, so this penalty term is proportional to the volatility of the portfolio.

The model considered in the previous section can also be generalized to handle multiple strategies at the same time. Indeed, one can think of an arbitrageur willing to take advantage of n misvaluations at the same time. In such a case, one introduces a vector $\boldsymbol{\varepsilon}_t$ governed by a vector Ornstein-Uhlenbeck process

$$d\boldsymbol{\varepsilon}_t = (-F\boldsymbol{\varepsilon}_t + G\dot{\boldsymbol{\varphi}}_t)dt + Sd\boldsymbol{W}_t \tag{7}$$

where F, G, and S are matrices with the obvious financial interpretations and the vector $\boldsymbol{\varphi}_t$ represents holdings in each trade. $\boldsymbol{W}$ is a d dimensional Brownian motion.

The problem faced by our arbitrageur is

$$\sup_{\{\boldsymbol{\varepsilon}_t\}} \mathbb{E}\left\{\int_0^T e^{-rt}\left[\boldsymbol{\varphi}_t(d\boldsymbol{\varepsilon}_t - r\boldsymbol{\varepsilon}_t dt) - \frac{1}{2}\dot{\boldsymbol{\varphi}}_t'\Lambda\dot{\boldsymbol{\varphi}}_t dt\right]\right\}$$

where Λ is a symmetric positive matrix representing the transaction costs.

The analysis and solution to this multi-dimensional problem is done in exactly the same way by introducing a $2n$-dimensional controlled diffusion. The solution is given in terms of a matrix Riccati equation where the unknown deterministic symmetric matrix is now of size $2n \times 2n$.

3. CONCLUSION

This paper discusses a simple model of market impacts and provides an explicit solution to the problem of computing optimal long/short trading strategies. Extensions to the multi-asset case are covered.

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PRICING EXCESS OF LOSS REINSURANCE WITH REINSTATEMENTS USING DERIVATIVE PRICING TECHNIQUES

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Abstract

This paper develops a framework for the pricing of excess of loss reinsurance contracts using derivative pricing techniques. The generalised Fourier transform method of Lewis (2001) is applied to provide an analytical formula for the expected cost to the reinsurer of an individual excess of loss treaty with any number of free or paid reinstatements. The advantage of this approach is that it does not require numerical approximations of the aggregate recovery distribution using Panjer recursion or alternative methods. Instead, direct computation can be carried out using efficient numerical integration techniques to evaluate the analytical pricing formula. The generalised Fourier transform methodology is illustrated through a numerical example using a Pareto claims severity distribution and is shown to provide accurate results by verification with Monte Carlo simulation.

1. INTRODUCTION

In this paper we will explore the application of modern derivative pricing techniques for the pricing of individual excess of loss (XL) reinsurance treaties. There are many natural analogies that can be drawn between reinsurance contracts and financial derivatives, which suggests that the financial pricing methodologies applied for option pricing could also play a valuable role in actuarial science. The connections between reinsurance and derivatives is clearly demonstrated by considering the form of many reinsurance contracts. For example, the payoff amount X for an aggregate excess of loss contract attaching at K with no limit and expiring at future time T on an underlying loss process S_t is

$$Z = \max(S_T - K, 0).$$

This has exactly the same form as the payoff function of a European call option with strike price K. Similarly, the total recoveries under an individual excess of loss reinsurance treaty attaching at

K with no limit and free reinstatements is given by

$$Z = \sum_{i=1}^{N_T} \max(X_i - K, 0),$$

where X_i is the amount of the ith claim and N_T is the total number of claims by time T. This again shows similarities to a derivative contract, in that each individual recovery has the form of an option payoff. We refer the reader to Embrechts (1996) and Holtan (2004) for a stimulating discussion of the connections between the actuarial and financial fields in this context.

This paper will focus on the specific problem of applying derivative pricing techniques to the valuation of individual excess of loss contracts with paid reinstatements. We will not consider the pricing of aggregate excess of loss reinsurance contracts and refer the reader to Haslip and Kaishev (2008) for a detailed investigation into this area. This paper is organised as follows:

In Section 2 we will introduce the generalised Fourier transform method and show how it can be used to provide market consistent pricing of derivative contracts under the framework provided by Lewis (2001).

Section 3 sets out the problem of pricing individual excess of loss contracts and derives an analytical expression for the price of a contract that provides unlimited free reinstatements.

Section 4 extends the pricing methodology to include individual excess of loss contracts with limited paid reinstatements by setting up an equation of value that compares the total recoveries to the total reinstatement premium and initial premium. This allows the correct level of initial premium to be determined.

Finally in Section 5 we provide a numerical example of applying the pricing framework. This considers an individual excess of loss contract that features limited paid reinstatements, with a Pareto claims severity distribution and Poisson claims frequency distribution.

2. GENERALIZED FOURIER TRANSFORM MODEL

The generalised Fourier transform method was introduced to financial mathematics by Lewis (2001) who demonstrated its use for option pricing under a Levy process. It provides a method for evaluating the expectation of a payoff function, which makes it ideal for pricing options and other contracts that depend on the value of a stochastic process at a future point in time. We define the generalised Fourier transform $\mathfrak{F}$ of a function $w : \mathbb{R} \to \mathbb{R}$ as

$$\mathfrak{F}\{w(x)\} = \hat{w}(\tau) = \int_{-\infty}^{\infty} e^{i\tau x} w(x) dx, \text{ where } \tau = u + iv.$$

The inverse generalised Fourier transform is defined as

$$\mathfrak{F}^{-1}\{\hat{w}(\tau)\} = w(x) = \frac{1}{2\pi} \int_{iv-\infty}^{iv+\infty} e^{-i\tau x} \hat{w}(\tau) d\tau,$$

where integration is performed along a straight line parallel to the real axis, along which τ stays within a strip of regularity.

The idea behind the generalised Fourier transform pricing method is to utilise the ability to switch the order of expectation and integration, when applying a consecutive Fourier and inverse Fourier transformation to the payoff function. Switching the order of expectation and integration is equivalent to switching the order of integration under a double integral. This is permissible under Fubini's theorem provided that the integrand is an $L^1(\mathbb{R}^2)$ function (see e.g. Weir (1973)). We will proceed under the assumption that this technical requirement is satisfied. In practice, this will normally be the case, since the usual choices of severity distributions such as the Gamma, Pareto and Log-Normal are well behaved.

To illustrate how the method works, consider a financial contract that provides a payoff of $\psi(S_T)$ at future time T, where S_t $(0 \leq t \leq T)$ is a stochastic process. For instance, S_t might represent the price of a stock and $\psi(S_T)$ could be the European option payoff function given by $\max(S_T - K, 0)$, where K is the contract strike price. The arbitrage free price V_0 at time 0 of such a financial contract is provided by the fundamental theorem of asset pricing as

$$V_0 = E_{\mathbf{Q}}\left\{e^{-rT}\psi(S_T)\right\}, \tag{1}$$

where r is the risk-free continuously compounded rate of interest and $\mathbf{Q}$ denotes that the expectation is calculated under a risk-neutral probability measure. We will now explain how the generalised Fourier transform method can be applied to evaluate this expectation. Consider the following algebraic manipulation

$$\begin{aligned}
V_0 &= E_{\mathbf{Q}}\left\{e^{-rT}\psi(S_T)\right\} = E_{\mathbf{Q}}\left\{e^{-rT}\mathfrak{F}^{-1}\left\{\mathfrak{F}\left[\psi(S_T)\right]\right\}\right\} \\
&= E_{\mathbf{Q}}\left\{\frac{e^{-rT}}{2\pi}\int_{iv-\infty}^{iv+\infty} e^{-i\tau S_T}\mathfrak{F}\left[\psi(S_T)\right]d\tau\right\} \\
&= \frac{e^{-rT}}{2\pi}\int_{iv-\infty}^{iv+\infty} E_{\mathbf{Q}}\left(e^{-i\tau S_T}\right)\mathfrak{F}\left[\psi(S_T)\right]d\tau \\
&= \frac{e^{-rT}}{2\pi}\int_{iv-\infty}^{iv+\infty} \phi_{S_T}(-\tau)\,\mathfrak{F}\left[\psi(S_T)\right]d\tau \\
&= \frac{e^{-rT}}{2\pi}\int_{iv-\infty}^{iv+\infty} \phi_{S_T}(-\tau)\,\hat{\psi}(\tau)d\tau
\end{aligned} \tag{2}$$

where ϕ_{S_T} is the characteristic function of S_T. The integral in (2) will generally be computed numerically along a strip parallel to the real axis in the complex plane within a region of regularity. This region will be determined by the $\tau \in \mathbb{C}$ for which $\mathfrak{F}[\psi(S_T)]$ is convergent and the characteristic function of S_T evaluated at $-\tau$ is analytical. The simplest way to evaluate this type of integral is to apply an efficient numerical integration algorithm. For instance, adaptive Gauss-Kronrod integration described by Calvetti et al. (2000) can be applied by making a substitution to remove the complex limits.

We will now consider how this financial pricing methodology can be applied to the problem of pricing individual excess of loss contracts. In doing so we will make the assumption that the claims distribution has been parameterised under a risk-neutral probability measure, so that we may apply the fundamental theorem of asset pricing in the context of reinsurance. We will not discuss the details of how one might select an appropriate risk-neutral measure, but refer the reader to Delbaen and Haezendonck (1989) for a discussion of how to derive risk-neutral probability measures corresponding to a number of actuarial premium principles.

3. INDIVIDUAL EXCESS OF LOSS

Here we consider the valuation of an individual excess of loss (XL) reinsurance treaty that applies to a specific insured class of risk, for example motor liability claims.

This type of reinsurance contract provides a payment to the cedent equal to the amount of each individual claim that exceeds a threshold limit L (the attachment point) and falls below an upper limit U. That is, for a claim of amount X, the reinsurer will pay $\min\left[\max\left(X-L,0\right),U-L\right]$ to the cedent. In normal reinsurance terminology this is referred to as a U-L XS L contract.

The period of cover under the reinsurance treaty will normally be one year and we denote the expiry time by T, where $T>0$. Additionally we assume that the reinsurer settles all payments at the end of the period of cover. The total recoveries Z_T at the expiry of the treaty are calculated as

$$Z_T=\sum_{i=1}^{N_T}\min\left[\max\left(X_i-L,0\right),U-L\right], \tag{3}$$

where each claim event occurs at time T_i, N_t is the number of claims that have occurred by time t and X_j is the severity of the jth claim. We follow the convention that $Z_T=0$ if $N_T=0$. The price of the reinsurance treaty at time 0 is therefore given by the fundamental theorem of asset pricing as

$$V_0=E_{\mathbf{Q}}\left(e^{-rT}Z_T\right)=E_{\mathbf{Q}}\left\{e^{-rT}\sum_{i=1}^{N_T}\min\left[\max\left(X_i-L,0\right),U-L\right]\right\}, \tag{4}$$

where $\mathbf{Q}$ denotes that expectation is to be carried out under a risk-neutral probability measure and r is the risk-free rate of interest. This treaty may be priced analytically if we assume that the individual claim amounts X_i are independent identically distributed random variables by following well known results for compound distributions.

For example, if we assume that claims arrive according to a Poisson process with arrival rate λ and the individual claim amount distribution is absolutely continuous with probability density function $f_X(x)$ then we can calculate the expected individual recovery per claim as

$$E_{\mathbf{Q}}\left(R_i\right)=E_{\mathbf{Q}}\left(\min\left[\max\left(X_i-L,0\right),U-L\right]\right)=\int_L^U(x-L)^+f_X(x)dx+P(X_i>U)(U-L),$$

where R_i denotes the individual recovery made on claim X_i. The aggregate expected recoveries for the reinsurance treaty under the risk-neutral probability measure will then be given by

$$E_{\mathbf{Q}}(Z_T) = E_{\mathbf{Q}}(N_T)E_{\mathbf{Q}}(R_i),$$

which follows from standard properties of compound distributions. For most choices of claim severity distribution this can be calculated to give a closed form analytical solution.

However, in practice most individual XL reinsurance treaties are not this simple. They normally will include clauses to limit the overall aggregate recovery allowable under the treaty and will only allow a specified number of reinstatements of the layer, for which additional premium may be payable. If a reinstatement premium is required this will be pre-specified in the treaty conditions at inception.

If we now assume that the total recoveries under the treaty are limited to amount M, with free reinstatements, then the total amount recoverable at expiry of the contract will be given by $\min(Z_T, M)$. The price of the reinsurance treaty therefore becomes

$$V_0 = E_{\mathbf{Q}}\left(e^{-rT}\min(Z_T, M)\right).$$

This is more difficult to calculate analytically and in practice Monte Carlo simulation is normally used for its evaluation. However, we will proceed to consider how this quantity could be calculated. Writing out the expectation in full it can be seen that

$$V_0 = e^{-rT}\int_0^{\infty}\min(z, M)f_{Z_T}(z)dz = e^{-rT}\left\{\int_0^{M} zf_{Z_T}(z)dz + MP(Z_T > M)\right\}, \tag{5}$$

which appears relatively simple, but requires knowledge of the probability density function f_{Z_T} of the aggregate recoveries process Z_T that was defined in (3). Several approaches exist to numerically approximate the aggregate distribution of compound processes, but these are computationally intensive and can be numerically imprecise. However, what we can more easily calculate is the characteristic function of Z_T. For example, under many popular choices for the claims arrival process, there are explicit formulae for the aggregate claims characteristic function. We will proceed without loss of generality by considering the case where the claims frequency is modelled by a Poisson distribution. A similar analysis can be carried out using alternative assumptions such as the Negative Binomial distribution.

If claims occur according to a Poisson process with arrival rate λ then the characteristic function of Z_T is given by

$$\phi_{Z_T}(\tau) = e^{\lambda T[\phi_R(\tau)-1]},$$

where $\phi_R(\tau) = E_{\mathbf{Q}}\left(e^{i\tau R}\right) = E_{\mathbf{Q}}\left(e^{i\tau\min[\max(X-L,0),U-L]}\right)$ is the characteristic function of the individual recovery amount R. We will now consider how to calculate the characteristic function of the individual recovery amount R, before proceeding to explain how this helps in calculating the price in (5).

We begin by noting that R has a mixed distribution: there is a mass of probability at $R = 0$ corresponding to claims that do not exceed the attachment point L and there is another probability mass at $R = U - L$ which corresponds to claims that exceed the upper limit U. When calculating the characteristic function of a random variable that has a mixed distribution, it often helps to first consider the probability mass function. By the law of total probability we have that

$$P(R \leq r) = \begin{cases} P(X < L) & \text{if } r = 0 \\ P(X < L) + P(L \leq X \leq r + L) & \text{if } 0 < r \leq U - L \\ 1 & \text{if } r > U - L \end{cases} . \tag{6}$$

This can be simplified to give

$$P(R \leq r) = \begin{cases} F_X(L) & \text{if } r = 0 \\ F_X(r + L) & \text{if } 0 < r \leq U - L \\ 1 & \text{if } r > U - L \end{cases} , \tag{7}$$

where $F_X(x)$ is the cumulative distribution function of the individual claim amount X. We now are able to calculate the characteristic function of R as

$$\begin{aligned} \phi_R(\tau) &= \int_{-\infty}^{\infty} e^{i\tau r} f_R(r) dr \\ &= e^0 P(R = 0) + \int_0^{U-L} e^{i\tau r} f_X(r + L) dr + e^{i\tau(U-L)} P(R = U - L) \\ &= F_X(L) + \int_0^{U-L} e^{i\tau r} f_X(r + L) dr + e^{i\tau(U-L)} \left[1 - F_X(U)\right]. \end{aligned} \tag{8}$$

As will be shown in Section 5, (8) is relatively simple to calculate and is tractable enough for efficient numerical implementation. The next step is to compute the generalised Fourier transform of $\min(Z_T, M)$. We begin by noting that since $Z_T \geq 0$, we have that $\min(Z_T, M) = \max[\min(Z_T, M), 0]$. Thus its Fourier transform $\hat{f}(\tau)$ is given by

$$\begin{aligned} \hat{f}(\tau) &= \mathfrak{F}\left(\max[\min(Z_T, M), 0]\right) = \int_{-\infty}^{\infty} e^{i\tau z} \max[\min(z, M), 0] dz \\ &= \int_0^M z e^{i\tau z} dz + M \int_M^{\infty} e^{i\tau z} dz \\ &= \frac{e^{i\tau M} - 1}{\tau^2}, \end{aligned} \tag{9}$$

where *Im*$(\tau) > 0$ is required for convergence.

We can now apply the technique developed by Lewis (2001) to calculate $E_{\mathbf{Q}}\left(\min(Z_T, M)\right)$.

$$\begin{aligned}
E_{\mathbf{Q}}\left(\min(Z_T, M)\right) &= E_{\mathbf{Q}}\left(\max[\min(Z_T, M), 0]\right) \\
&= E_{\mathbf{Q}}\left\{\mathfrak{F}^{-1}\left(\frac{e^{i\tau M}-1}{\tau^2}\right)\right\} \\
&= E_{\mathbf{Q}}\left\{\frac{1}{2\pi}\int_{iv-\infty}^{iv+\infty} e^{-i\tau Z}\left(\frac{e^{i\tau M}-1}{\tau^2}\right)d\tau\right\} \\
&= \frac{1}{2\pi}\int_{iv-\infty}^{iv+\infty} E_{\mathbf{Q}}\left(e^{-i\tau Z}\right)\left(\frac{e^{i\tau M}-1}{\tau^2}\right)d\tau \\
&= \frac{1}{2\pi}\int_{iv-\infty}^{iv+\infty} e^{\lambda T[\phi_R(-\tau)-1]}\left(\frac{e^{i\tau M}-1}{\tau^2}\right)d\tau, \qquad (10)
\end{aligned}$$

where we require *Im*$(\tau) > 0$. This means that integration must be carried out on a straight line that lies above the real axis where $v > 0$. It is noted that this integral can be evaluated efficiently using a number of numerical integration techniques that are described in Haslip and Kaishev (2008).

4. REINSTATEMENTS

As explained in the previous section, most individual XL treaties include provisions for a number of reinstatements of the layer for a pre-specified premium. In this section we will consider how to price a contract that offers a limited number of reinstatements, each at a fixed premium. The way reinstatements operate, is that after notification of each individual claim that generates a recovery, an additional reinstatement premium must be paid in order to reinstate the layer to its full level. Normally the reinstatement premium is expressed as a percentage of the original reinsurance premium. If we denote the original reinsurance premium by p and the reinstatement premium payable for each reinstatement as $p_1, p_2, ..., p_{n-1}$ then the amount of reinstatement premium payable after a claim leading to a recovery R is given by

$$p\left\{R_I p_m + R_O p_{m+1}\right\},$$

where $m = \lceil \frac{Z}{U-L} \rceil$ is the integer $+\ 1$ of the argument, which provides number of reinstatements required once the current layer is exhausted. $R_I = \min(m(U-L) - Z, R)$ is the amount of the recovery falling into the current reinstatement band and $R_O = \max(R - R_I, 0)$ is the amount of the recovery falling into the next band of reinstatements.

The total reinstatement premium (TRP) received at the end of the period of cover can therefore be calculated as

$$\text{TRP} = p\sum_{i=1}^{n-1}\frac{p_i}{U-L}\min\left[\max\left(Z-(i-1)(U-L), 0\right), U-L\right], \qquad (11)$$

where it is assumed that premiums are settled at the end of the period of cover in aggregate. This can be seen by considering that each rate of reinstatement premium will apply to all recoveries within the corresponding band of reinstatements. For instance, reinstatement premium rate $\frac{p_i}{U-L}$ will apply to recoveries between $(i-1)(U-L)$ and $i(U-L)$. Therefore in order to calculate the expected total reinstatement premium receivable, we need to only be able to calculate $E_{\mathbf{Q}}\{\min[\max(Z-(i-1)(U-L),0),U-L]\}$. The Fourier transform $\hat{f}(\tau)$ of $\min[\max(Z-A,0),B-A]$ is given by

$$\begin{aligned}\hat{f}(\tau) &= \mathfrak{F}(\min[\max(Z-A,0),B-A]) = \int_A^B e^{i\tau z} z dz + (B-A)\int_B^\infty e^{i\tau z} dz \\ &= \frac{e^{i\tau B}-e^{i\tau A}}{\tau^2},\end{aligned}$$

where we require $Im(\tau) > 0$ for convergence. Using the generalised Fourier transform pricing approach, we can therefore calculate the expected recoveries falling into each band of reinstatements as

$$E_{\mathbf{Q}}\{\min[\max(Z-A,0),B-A]\} = \frac{1}{2\pi}\int_{iv-\infty}^{iv+\infty} e^{\lambda T[\phi_R(-\tau)-1]}\left(\frac{e^{i\tau B}-e^{i\tau A}}{\tau^2}\right)d\tau, \quad (12)$$

where $A=(i-1)(U-L)$ and $B=i(U-L)$ and integration is carried out along a straight line that lies above the real axis in a strip of regularity by choosing a suitable $v>0$. We therefore are now able to calculate $E_{\mathbf{Q}}(\text{TRP})$ and can proceed to consider how to set the correct level of initial premium, in the presence of limited paid reinstatements. In setting the correct initial premium we consider the following equation of value

$$E_{\mathbf{Q}}[\min(Z,n(U-L))] = p\left[1+\sum_{i=1}^{n-1}\frac{p_i}{U-L}E_{\mathbf{Q}}\{\min[\max(Z-(i-1)(U-L),0),U-L]\}\right], \quad (13)$$

where we allow $n-1$ reinstatements and an initial free reinstatement that is provided by the initial premium. Since the required reinstatement premium percentages are fixed, we can solve (13) for p to calculate the required initial premium. We note that Mata (2000) provides a similar expression for the calculation of the initial reinsurance premium under a pure premium framework.

5. EXAMPLE

To illustrate this methodology we will consider an example of pricing an individual excess of loss treaty with limited paid reinstatements. It is assumed that under the risk-neutral probability measure, losses arrive according to a Poisson process with rate $\lambda=3$ per annum and claim severity follows a type II Pareto distribution with parameters $k=2$, $\alpha=0.5$. The Pareto probability density function is given by $f_X(x)=\frac{k\alpha^k}{x^{k+1}}$ and the cumulative distribution function is $F_X(x)=1-\left(\frac{\alpha}{x}\right)^k$. We will consider pricing an individual excess of loss treaty that provides recoveries on individual losses between $L = 1$ and $U = 2$ over a 1 year time horizon and allows a maximum of 3 reinstatements at rates $p_1 = 100\%$, $p_2 = 75\%$ and $p_3 = 50\%$ of the initial premium p. We begin by calculating the characteristic function of the individual recovery amount $R = \min[\max(X - L, 0), U - L]$. Applying (8) we have

$$\begin{aligned}\phi_R(\tau) &= 1 - \left(\frac{\alpha}{L}\right)^k + \int_0^{U-L} e^{i\tau r} \frac{k\alpha^k}{(r+L)^{k+1}} dr + e^{i\tau(U-L)} \left(\frac{\alpha}{U}\right)^k \\ &= 1 - \left(\frac{\alpha}{L}\right)^k + e^{-i\tau L} k(-i\alpha\tau)^k \int_{-iL\tau}^{-iU\tau} e^{-y} y^{-k-1} dy + e^{i\tau(U-L)} \left(\frac{\alpha}{U}\right)^k \\ &= 1 - \left(\frac{\alpha}{L}\right)^k + e^{-i\tau L} k(-i\alpha\tau)^k \left[\Gamma(-k, -iL\tau) - \Gamma(-k, -iU\tau)\right] + e^{i\tau(U-L)} \left(\frac{\alpha}{U}\right)^k\end{aligned}$$

where Γ is the upper incomplete Gamma function defined by $\Gamma(a, x) = \int_x^\infty t^{a-1} e^{-t} dt$.

The next step is to compute the expectations required in (13) to solve for the initial premium. This means that we should calculate the following quantities

1. $E_0 = E_{\mathbf{Q}}\left(\min\left[Z, 4(U-L)\right]\right) = E_{\mathbf{Q}}\left(\min\left[Z, 4\right]\right)$
2. $E_1 = E_{\mathbf{Q}}\left(\min\left[Z, U-L\right]\right) = E_{\mathbf{Q}}\left(\min\left[Z, 1\right]\right)$
3. $E_2 = E_{\mathbf{Q}}\left(\min\left[\max\left(Z - (U-L), 0\right), U-L\right]\right) = E_{\mathbf{Q}}\left(\min\left[\max\left(Z - 1, 0\right), 1\right]\right)$
4. $E_3 = E_{\mathbf{Q}}\left(\min\left[\max\left(Z - 2(U-L), 0\right), U-L\right]\right) = E_{\mathbf{Q}}\left(\min\left[\max\left(Z - 2, 0\right), 1\right]\right)$.

We can then solve the following equation of value to determine the correct initial premium p

$$E_0 = p\left[1 + E_1 + 0.75E_2 + 0.5E_3\right].$$

To determine E_0, E_1, E_2, E_3 we can apply formulae (10) and (12) using numerical integration. This can be carried out using any standard mathematical package or implemented in a programming language using specialised algorithms for greater numerical efficiency. For simplicity in this paper, we have performed the numerical integrations using the `NIntegrate` function in Mathematica with $v = 0.5$. This yielded the following results

$$\begin{aligned}E_0 &= 0.375145928700 \\ E_1 &= 0.319450478900 \\ E_2 &= 0.050310246860 \\ E_3 &= 0.005023995497\end{aligned}$$

and hence we find $p = 0.275904$. This premium calculation has been verified by applying Monte Carlo simulation for 10 million trials which yielded an initial premium within 0.1% of this calculation.

6. CONCLUSION

In this paper we have demonstrated how derivative pricing techniques can be successfully applied to the problem of pricing individual excess of loss reinsurance treaties.

This work has some common themes with Mata (2000), in that it essentially is concerned with the calculation of pure premium via expectations. The key difference is that rather than applying Panjer recursion methods, we have demonstrated a direct analytical formula for the expectations that can be computed to a high degree of accuracy in a single computation. An interesting extension of this paper would be to apply the generalised Fourier transform methodology to other premium principles under a real world probability measure. For instance, the method described in this paper could be applied to calculate the variance of the contract payoff, thus allowing the standard deviation premium principle to be applied. In this sense, the generalised Fourier transform pricing method can be seen as an alternative to Monte Carlo simulation for actuarial pricing exercises.

One important consideration in the numerical implementation of the method proposed in this paper, is the choice of the complex shift parameter v. Numerical studies in Haslip and Kaishev (2008) for pricing catastrophe bonds and aggregate excess of loss contracts, indicate that the pricing integral converges quickly, for normal choices of attachment point and time to expiry. However, for attachment points closer to zero and contracts with short maturity times, the pricing integral becomes oscillatory and difficult to compute. This problem is well documented in the context of pricing financial options under the generalised Fourier transform method. Lord and Kahl (2006) describe a method to identify the choice of v that provides optimal convergence, given a particular strike price and expiry date. Their approach could be adapted to find the optimal value of v for pricing reinsurance contracts.

Finally, to assess the practical value of the approach described in this paper, it would be valuable to perform a comparison of its numerical efficiency for pricing individual excess of loss contracts compared to other methods such as Monte Carlo simulation, Panjer recursion and the numerical solution of partial differential equations.

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APPENDIX: COMPUTATION OF NUMERICAL INTEGRALS

In order to evaluate (10) and (12) we have utilised the `NIntegrate` function in Mathematica. Since (10) can be calculated as a special case of (12) where $A = 0$ and $B = M$, we only needed to implement the second formula. This has been achieved by defining a function `reinstate` (A, B) as shown in the following Mathematica code:

```
phiULInd[z_] :=
 k (-\[ImaginaryI]*\[Alpha]*z)^k*(Gamma[-k, -\[ImaginaryI]*L*z]
 - Gamma[-k, -\[ImaginaryI]*U*z])*Exp[-\[ImaginaryI]*z*L]
 + Exp[\[ImaginaryI]*z*(U - L)]*((\[Alpha]/U)^k) + 1 - (\[Alpha]/L)^k

reinstate[A_, B_] :=
 N[NIntegrate[1/ (2*\[Pi]) *((Exp[\[ImaginaryI]*z*B]
 - Exp[\[ImaginaryI]*z*A]) / z^2) * Exp[\[Lambda]*(phiULInd[-z] -  1)],
 {z, -900 + 0.5 \[ImaginaryI], 900 + 0.5 \[ImaginaryI]},
 PrecisionGoal -> 10, WorkingPrecision -> 90], 10];
```

SINGLE NAME CREDIT DEFAULT SWAPTIONS MEET SINGLE SIDED JUMP MODELS [1]

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Abstract

Credit risk modeling is about modeling losses. These losses are typically coming unexpectedly and triggered by shocks. So any process modeling the stochastic nature of losses should reasonably include jumps. In this paper we review a few jump driven models for the valuation of CDSs and show how under these dynamic models also pricing of (exotic) derivatives on single name CDSs is possible. More precisely, we set up fundamental firm-value models that allow for fast pricing of the 'vanillas' of the CDS derivative markets: payer and receiver swaptions. It turns out that the proposed model is able to produce realistic implied volatility smiles. Moreover, we detail how a CDS spread simulator can be set up under this framework and illustrate its use for the pricing of exotic derivatives on single name CDSs as underliers.

1. INTRODUCTION

Credit Default Swaps (CDSs) have become in the last decennium very important instruments to deal with credit risk. These financial contracts are now available in quite liquid form on thousands of underliers and are traded daily in huge volume. A market has been formed dealing with options or derivatives on these CDSs. The market is for the moment quite illiquid, but is expected to gain in volume over the next years.

Credit risk modeling is about modeling losses. These losses are typically coming unexpectedly and triggered by shocks. So any process modeling the stochastic nature of losses should reasonably include jumps. The presence of jumps is even of greater importance if one deals with derivatives on CDSs, because of the leveraging effects. Jump processes have proven already their modeling abilities in other settings like equity and fixed income (see Schoutens (2003)) and have recently found their way into the credit risk modeling. In this paper we review a few jump driven models for the valuation of CDSs and show how under these dynamic models also pricing of (exotic)

[1] This is a short version of Jönsson and Schoutens (2007).

derivatives on single name CDSs is possible. More precisely, we set up fundamental firm-value models that allow for fast pricing of the 'vanillas' of the CDS derivative markets: payer and receiver swaptions. Moreover, we detail how a CDS spread simulator can be set up under this framework and illustrate its use for the pricing of exotic derivatives on single name CDSs as underliers.

The standard available model to price swaptions is Black's model and implied volatilities are often extracted out of market data completely similar as in the classical Black-Scholes equity model. In the scarce available market data a volatility smile is typically present. The model proposed here is able to produce such a volatility smile.

The paper is organized as follows. In Section 2, we present the one sided Lévy firm value model and give one example of a possible choice of Lévy processes to use, namely the Gamma process. In Section 3, we discuss the pricing of receiver and payer swaptions, by mapping the firm value to the spread value. We show how dynamic spreads can be generated by mapping the firm value paths to spread paths and how this can be used to price exotic options by Monte Carlo techniques in Section 4. Section 5 deals with the pricing of American options.

A full length version of this text is published in Jönsson and Schoutens (2007).

2. FIRM VALUE MODELS

The starting point of the model is the approach originally presented by Black and Cox (1976). According to this approach an event of default occurs when the asset value of the firm crosses a deterministic barrier. This barrier corresponds to the recovery value of the firm's debt.

Black and Cox assumed a geometric Brownian motion for the firm's value processes. It is however well known that due to the continuous path nature of Brownian motion and the fast decaying tails of the underlying Normal distribution, the model cannot represent a realistic behavior of short term default probabilities. Indeed, the Brownian motion needs a substantial amount of time to reach a low barrier. In order to overcome these kind of shortcomings, many extensions were already proposed. The CreditGrades$^{\text{TM}}$ (2002) model for example tried to lift the short time default probabilities to make the barrier stochastic.

Here we use the same methodology as Black and Cox but work under exponential Lévy models; Default is triggered when the firm's value is crossing a low-barrier. By doing that, due to the jump nature, not only realistic default probabilities could be produced but also the other problem of predictability of the default time is overcome. Different Lévy models (both firm value and intensity) for credit derivatives pricing were explored by Cariboni (2007).

2.1. One Sided Lévy Processes

We first introduce some notation. Let $Y = \{Y_t, t \geq 0\}$ be a pure jump Lévy process that has only negative jumps, that is, Y is spectrally negative, and let $X = \{X_t, t \geq 0\}$ be given by

$$X_t = \mu t + Y_t, \quad t \geq 0,$$

where μ is positive real number.

The Laplace transform of X_t

$$\mathbb{E}[\exp(zX_t)] = \exp(t\psi_X(z)),$$

where $\psi_X(z)$ is the Lévy exponent, which by the Lévy-Khintchin representation has the form

$$\psi_X(z) = \mu z + \int_{-\infty}^{0} (e^{zx} - 1 + z(|x| \wedge 1))\nu(dx).$$

The Lévy measure $\nu(dx)$ satisfies the integrability condition

$$\int_{-\infty}^{0} (|x| \wedge 1)\nu(dx) < \infty.$$

For the processes we consider in this paper the Lévy measure has a density and we can write $\nu(dx) = m(x)dx$, where $m(x)$ is the density function. For the general theory of Lévy processes see, for example, Bertoin (1996) and Sato (2000).

2.2. Lévy Firm Value Models

We thus work under a firm's value setting, where the fundamental process that we model is the value of the reference entity of a CDS, and we opt to model the firm's value by exponential Lévy driven jump models. These Lévy models have proven their modeling abilities already in different fields, but especially in the credit modeling setting we are absolutely convinced of the necessity of jumps in the modeling of the fundamental underlying process.

So assume, $X = \{X_t, t \geq 0\}$ is a pure jump Lévy process. The (risk neutral) value of the firm at time t is then modeled by

$$V_t = V_0 \exp(X_t), \quad t \geq 0,$$

and we work under an admissible pricing measure $\mathbb{Q}$ such that $\mathbb{E}_{\mathbb{Q}}[V_t] = V_0 \exp(rt)$, where r is the risk-free interest rate. We will refer to the process V as the firm's value process. For a given recovery rate R, default occurs the first time

$$V_t = V_0 \exp(X_t) \leq RV_0 \ ,$$

or, equivalently, if

$$X_t \leq \log R.$$

Let us denote by $P(t)$ the risk-neutral *survival probability*, or in other words the probability of no-default, between 0 and t:

$$\begin{aligned} P(t) &= \mathbb{P}_{\mathbb{Q}}\left(\min_{0\leq s\leq t} X_S > \log R\right) \\ &= \mathbb{E}_{\mathbb{Q}}\left[\mathbf{1}\left(\min_{0\leq s\leq t} V_s > RV_0\right)\right], \end{aligned}$$

where we used the indicator function $\mathbf{1}(A)$, which is equal to 1 if the event A is true and zero otherwise; the subindex $\mathbb{Q}$ refers to the fact that we are working in a risk-neutral setting.

Consider a CDS with maturity T and a continuous spread c. The price of this CDS is then given by

$$CDS = (1-R)\left(-\int_0^T \exp(-rs)\mathrm{d}P(s)\right) - c\int_0^T \exp(-rs)P(s)\mathrm{d}s,$$

where R is the asset specific recovery rate and r is the default-free discount rate. Note that in case of a default event the protection buyer is receiving $(1-R)$ for every insured currency unit. From this, we find the par spread C that makes the CDS price equal to zero:

$$C = \frac{(1-R)\left(-\int_0^T \exp(-rs)\mathrm{d}P(s)\right)}{\int_0^T \exp(-rs)P(s)\mathrm{d}s},$$

where the denominator is the *risky annuity*, that is, the value of a one basis point premium leg.

During the life-time of the CDS, the protection seller receives the fair par spread C on the insured amount as a compensation for the default risk taken on.

As we can see the pricing of a CDS depends fully on the expression of the default probability of the firm. In recent years, the calculation of these default probabilities (or equivalently in this setting, the hitting time probabilities) under several Lévy driven models have been worked out. We mention especially two approaches where the calculations can be done very fast, namely the Partial Integral-Differential Equation (PIDE) approach and the double Laplace inversion approach based on the Wiener-Hopf Factorization. The first one can deal with general Lévy processes and is worked out for the very popular Variance Gamma (VG) case in for example Cariboni and Schoutens (2007), the second approach is only tractable for spectrally one-sided processes, only allowing for negative jumps and can be found in Rogers (2000) and Madan and Schoutens (2007). However, one could argue that in contrast to stock price behavior where clearly up and down jumps are present, a firm tries to follow a steady growth (up trend) but is exposed to shocks (negative jumps). It thus seems quite natural to model the underlying firm's value in a default model by a process with a positive drift and allow only for negative jumps. In contrast to the double sided situation, where the solution of the PIDE takes typically a couple of seconds on an ordinary computer, the double Laplace inversion can be performed within a fraction of a second.

2.3. Example - The Shifted Gamma-Model

We present here the one well known example of Lévy model with positive upward trend and negative jumps - the Shifted Gamma model. Two other models, the Shifted Inverse Gaussian and the Shifted CMY, are described in Madan and Schoutens (2007) and Jönsson and Schoutens (2007).

The density function of the Gamma distribution Gamma(a,b) with parameters $a>0$ and $b>0$ is given by

$$f_{Gamma}(x;a,b) = \frac{b^a}{\Gamma(a)} x^{a-1}\exp(-xb), \quad x>0.$$

The characteristic function is given by

$$\phi_{Gamma}(u;a,b) = (1-\mathrm{i}u/b)^{-a}, \quad u\in\mathbb{R}.$$

Clearly, this characteristic function is infinitely divisible. The Gamma-process $G = \{G_t, t \geq 0\}$ with parameters $a, b > 0$ is defined as the stochastic process which starts at zero and has stationary, independent Gamma-distributed increments. More precisely, the time enters in the first parameter: G_t follows a Gamma(at, b) distribution.

The Lévy density of the Gamma process is given by

$$m(x) = a \exp(-bx)x^{-1}, \quad x > 0.$$

The properties of the Gamma(a, b) distribution given in Table 1 can easily be derived from the characteristic function.

	Gamma(a, b)
mean	a/b
variance	a/b^2
skewness	$2/\sqrt{a}$
kurtosis	$3(1 + 2/a)$

Table 1: Mean, variance, skewness and kurtosis of the Gamma distribution.

Note also that we have the following scaling property: if X is Gamma(a, b) then for $c > 0$, cX is Gamma$(a, b/c)$.

Let us start with a unit variance Gamma-process $G = \{G_t, t \geq 0\}$ with parameters $a > 0$ and $b > 0$. As driving Lévy process (in a risk-neutral setting), we then take

$$X_t = \mu t - G_t, \quad t \geq 0,$$

where in this case $\mu = r - \log(\phi(i)) = r + a \log(1 + b^{-1})$. Thus, there is a deterministic up trend with random downward shocks coming from the Gamma process.

The characteristic exponent is in this case available in closed form

$$\psi(z) = \mu z - a \log(1 + zb^{-1}).$$

Calibrating the Shifted Gamma model to the term structure of BAE Systems on the 5th of January 2005 gives the parameters $a = 1.2028$ and $b = 5.9720$. The fit of the Shifted Gamma model on the market CDSs is shown in Figure 1.

An extensive calibration study was performed by Madan and Schoutens (2007) and the fitting error was typically around 1-2 basis points per quote.

3. OPTIONS ON CDS

The firm's value approach is ideally suitable to set up a methodology for the pricing of options on CDSs. The most common options are payer and receiver swaptions. They are the counterpart of the classical European call and put options in other markets.

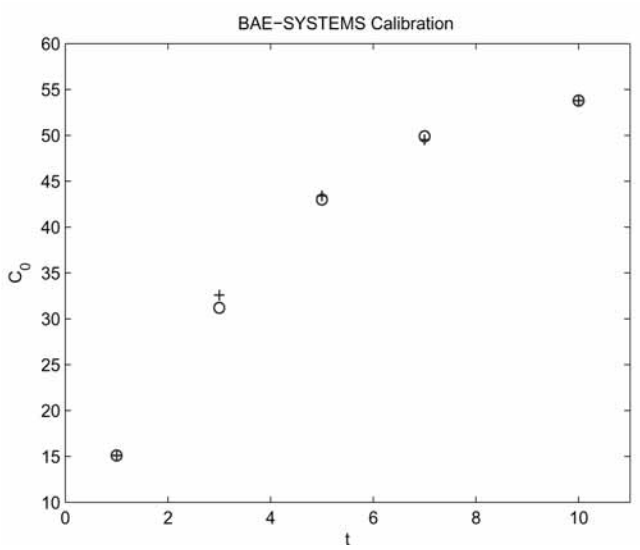

Figure 1: Calibration BAE Systems. Underlying model is the Shifted Gamma with $a = 1.2028$ and $b = 5.9720$.

A receiver option holder on a single name CDS has the right to sell protection at the strike spread level on expiry, in return for buying protection on the same CDS at market spread level on expiry. A payer option holder on a single name CDS has the right but not the obligation to buy protection on the underlying CDS at the strike level on expiry, in return for selling protection at par spread at expiry. Special rules apply for the case when default happens before the option's maturity. Most common situation on the single-name payer and receiver structures is a knock-out clause, meaning that in case of early default the payoff is zero.

3.1. Swaption Valuation

Suppose we have a particular (Lévy driven) firm's value model under consideration and want to price options on CDS under that model. As explained above the par spread under such a model depends on the following parameters:

- the initial firm's value, V_0;
- the recovery rate R, which also implies the default barrier V_0R;
- the risk-free rate r;
- the time of maturity of the CDS, T;
- the parameters of the underlying Lévy process, which we denote by the vector θ.

Hence let us write for the fair par spread (at time zero):

$$C_0 = C(V_0, R, r, T, \theta).$$

Similarly we write:

$$CDS_0(V_0, R, r, T, \theta, c)$$

for the price to enter into a CDS agreement at time zero for a contract paying c premium. Note that $CDS_0(V_0, R, r, T, \theta, C_0) = 0$.

Consider a European option on a CDS and denote the maturity of this option by T^*. For the standard examples of the payer and receiver option, the payoff depend on the spread value at T^* and in general the payoff can depend on the full evolution of the spread until the option's maturity, T^*. Hence, it is important to have a (stochastic) model for the spread at a future time point or more general on the evolution of the spread over a given time frame.

Such a model can be readily set up under a firm's value approach by mapping it back to the evolution of the firm's value. More precisely, we will propose the following stochastic model for the spread evolution over time:

$$C_t = C(V_t, R, r, T, \theta), \quad t \geq 0.$$

Similarly, we have also the evolution of the upfront price:

$$CDS_t(c) = CDS_t(V_t, R, r, T, \theta, c), \quad t \geq 0.$$

Again, we have

$$CDS_t(C_t) = CDS_t(V_t, R, r, T, \theta, C_t) = 0.$$

We will ignore in the modeling of the par spread the situation of early default and take this immediately into account in the options payoff.

Let us write $F(\{CDS_t, 0 \leq t \leq T^*\}, \mathbf{1}(\tau < T^*))$ for the payoff function to indicate that the payoff can depend on the full path and on whether default occurred early $\mathbf{1}(\tau < T^*)$ or not.

The price of an option is by risk-neutral valuation theory given by

$$\Pi_0 = \exp(-rT^*)\mathbb{E}_{\mathbb{Q}}[F(\{CDS_t, 0 \leq t \leq T^*\}, \mathbf{1}(\tau < T^*))] .$$

For a knock-out receiver and payer with strike spread K this simplifies to respectively:

$$\begin{aligned}\Pi_0^R(T^*, K) &= \exp(-rT^*)\mathbb{E}_{\mathbb{Q}}[(CDS_{T^*}(V_{T^*}, R, r, T, \theta, C_{T^*}) \\ &\qquad -CDS_{T^*}(V_{T^*}, R, r, T, \theta, K))^+\mathbf{1}(\tau > T^*)] \\ &= \exp(-rT^*)\mathbb{E}_{\mathbb{Q}}[(-CDS_{T^*}(V_{T^*}, R, r, T, \theta, K))^+|\tau > T^*]P(T^*) ;\end{aligned}$$

$$\begin{aligned}\Pi_0^P(T^*, K) &= \exp(-rT^*)\mathbb{E}_{\mathbb{Q}}[(CDS_{T^*}(V_{T^*}, R, r, T, \theta, K) \\ &\qquad -CDS_{T^*}(V_{T^*}, R, r, T, \theta, C_{T^*}))^+\mathbf{1}(\tau > T^*)] \\ &= \exp(-rT^*)\mathbb{E}_{\mathbb{Q}}[(CDS_{T^*}(V_{T^*}, R, r, T, \theta, K))^+|\tau > T^*]P(T^*) .\end{aligned}$$

Cleverly studying and implementing the PIDE algorithm for the general Lévy case or the double Laplace inversion method for spectrally negative processes, learns that it is actually not really computationally more intensive to calculate $CDS(V_0 \exp(x), R, r, T, \theta, K)$ for a whole range of x's.

In Figure 2 one finds the prices for 1 year receiver options on the BAE Systems CDS with 5 year maturity for a whole range of strike spreads calculated using the Shifted Gamma model with parameters $a = 1.2028$ and $b = 5.9720$ given by calibrating the model to the BAE Systems spread term structure on January 5, 2005. The spot (par) spread and the forward spread was $C_0 = 43$ and 47 basis points, respectively.

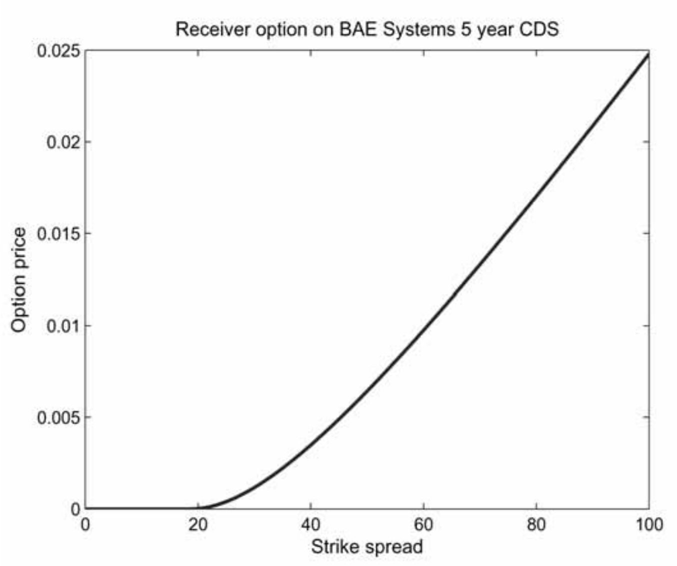

Figure 2: One year receiver option prices - Shifted Gamma model - BAE Systems CDS with 5 year maturity. Underlying model is the Shifted Gamma with $a = 1.2028$ and $b = 5.9720$. The spot (par) spread was $C_0 = 43$ basis points and forward spread of 47 bp.

3.2. Black's Formulas and Implied Volatility

The market standard for pricing options on CDSs are the Black formulas, based on the assumption that the credit spread follows a Geometric Brownian motion.

Before we write down the formulas, let us first introduce some notation. We denote by $F_0(T^*, T)$ the forward spread and $A_0(T^*, T)$ the forward starting risky annuity at $t = 0$ from swaption maturity T^* to CDS maturity T of a forward starting CDS. A forward starting CDS is an agreement today to enter into a CDS contract at a future time.

We can now write down the Black formulas for payer and receiver swaptions respectively

$$Payer_0(T, K) = A_0(T^*, T)(F_0(T^*, T)N(d_1) - KN(d_2))$$

$$Receiver_0(T, K) = A_0(T^*, T)(KN(-d_2) - F_0(T^*, T)N(-d_1)),$$

where

$$d_1 = \frac{\log(F_0(T^*, T)/K) + \sigma^2 T^*/2}{\sigma\sqrt{T^*}} \text{ and } d_2 = d_1 - \sigma\sqrt{T^*}.$$

Given the forward spread and risky annuities we can calculate the implied volatilities for payers and receivers with different strikes. Implied volatilities for payers and receivers written on the same underlying CDS are given in Table 2. We have assumed a flat term structure of the interest rates of 3%. The values of the options are generated using Shifted Gamma model ($a = 1.2028$ and $b = 5.9720$).

Strike (bp)	Payer	Implied vol ($\%$)	Receiver	Implied vol ($\%$)
40.0	0.003710	42.4	0.001025	57.9
42.0	0.003414	52.0	0.001636	65.3
44.0	0.003164	59.1	0.002293	71.4
46.0	0.002948	64.8	0.002984	76.6
48.0	0.002758	69.7	0.003702	81.3
50.0	0.002589	73.9	0.004440	85.4

Table 2: The estimated values of an European payer and receiver, respectively, with maturity 0.25 year to enter into a single name CDS (BAE Systems) with a five year maturity and the corresponding Black's implied volatilities. The forward spread is 47 bp. The parameters of the underlying Shifted Gamma model are $a = 1.2028$ and $b = 5.9720$.

4. A CDS SPREAD GENERATOR AND THE PRICING OF EXOTIC OPTIONS ON CDS

Using the same methodology of mapping spreads and firm's values, it is straightforward to generate spreads if a generator is available for the firm's value process.

In order to implement the method, first calibrate the model on a given term-structure of market spreads in order to find the model parameters θ that matches best the current market situation. Next, we precalculate spread values $\{C(V_0 \exp(x_i), R, r, T, \theta), i = 1, \ldots, n\}$ for a fine grid $\{x_1, ..., x_n\}$. Once this is done we start generating paths of the firm's value process. For each firm value $V_t = V_0 \exp(X_t)$, the corresponding spread is obtained by interpolating in X_t the grid $\{x_1, ..., x_n\}$ and its corresponding function values $\{C(V_0 \exp(x_i), R, r, T, \theta), i = 1, \ldots, n\}$.

In Figure 3, we see a path of a firm's value process under the Shifted Gamma model and its corresponding spread evolution. The parameters of the Gamma model are obtained by a calibration of the spread structure of BAE Systems on the January 5, 2005 ($a = 1.2028$ and $b = 5.9720$).

Once a fast spread generator is implemented we are set to price by Monte Carlo methods all kinds of exotic European structures on the evolution of the spread of a single name CDS.

We illustrate this by pricing a knock-out Variance Swap ($T^* = 1$, $t_i = i/252$, $i = 0, ..., 252$) :

$$\text{VS} = \exp(-rT^*)\mathbb{E}_{\mathbb{Q}}\left[\sum_{n=1}^{252}(\log(C_{t_i}) - \log(C_{t_{i-1}}))^2 | \tau > T_*\right] P(T_*).$$

The corresponding price under $100,000$ Monte-Carlo simulations is 0.00187 and was obtained on an ordinary PC in less than 5 minutes when implemented in MATLAB.

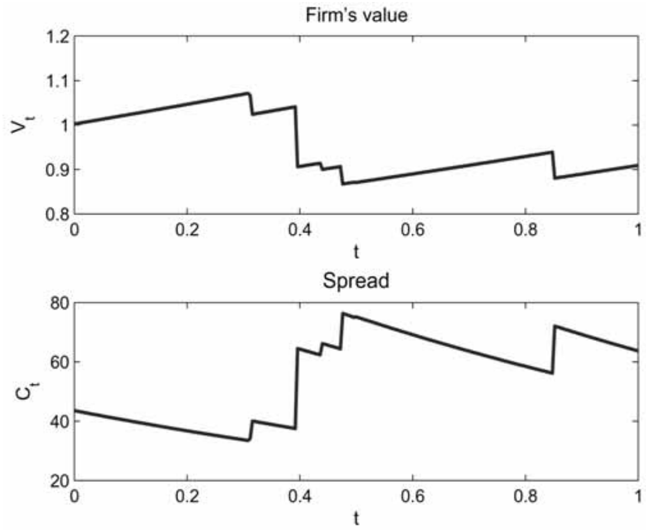

Figure 3: Spread path - BAE Systems. Underlying model is the Shifted Gamma with $a = 1.2028$ and $b = 5.9720$. Spot par spread $C_0 = 43$.

5. REGRESSION BASED PRICING OF BERMUDAN SWAPTIONS

With the approach to generate spread paths described above it is also possible to price American and Bermudan swaptions using the same methods as for options written on equities. For example, the regression based algorithms by Longstaff and Schwartz (2001) and Tsitsiklis and Van Roy (1999, 2001) can be adjusted to take into account the possibility that the option knock-out at the event of a default before option maturity and between exercise dates. An extensive overview of Monte Carlo methods in finance is given by Glasserman (2004).

Values for an European payer and Bermudan payers with semiannually, quarterly and monthly exercise opportunities, respectively, all with maturity of one year, to enter into a CDS with a fixed spread and five years maturity are shown in Table 3 for different strike spreads.

We used weighted Laguerre polynomials as a function of the spread value plus a constant as the basis functions for the regression. We could use other basis functions, such as powers of the state variable. We have however not noticed any difference in accuracy comparing the values and standard errors using different basis functions.

For a set of $100,000$ paths it takes approximately 25, 30 and 60 minutes to price the semiannual, the quarterly and the monthly Bermudan option, respectively, for the range of strikes given in Table 3, when implemented in MATLAB.

To test the algorithm we have used the diagnostic test proposed in Longstaff and Schwartz (2001), namely, we first do the least squares estimation of the regression parameters based on one sample of paths and then use these estimates to price the option using a second sample of paths. The in-sample and out-of-sample option values do not differ significantly and the standard errors are of the same magnitude.

Strike (bp)	European	Ber. semiann.	Ber. quarterly	Ber. monthly
44	0.009299	0.010305	0.012064	0.017081
46	0.009007	0.009854	0.011290	0.015628
48	0.008732	0.009450	0.010728	0.014597
50	0.008473	0.009114	0.010256	0.013955
52	0.008229	0.008802	0.009818	0.013075
54	0.007999	0.008505	0.009428	0.012436
56	0.007780	0.008223	0.009011	0.011781

Table 3: The estimated values of an European payer and Bermudan payers with semiannual, quarterly and monthly exercise, respectively, all with maturity one year to enter into a single name CDS with a five year maturity. The least squares estimate of the continuation value is based on the spread value. The simulation is based on 100,000 paths for the firm value process with 250 discretization points per year. The standard error of the estimates all lies between $0.86 \cdot 10^{-4}$ and $1.25 \cdot 10^{-4}$, with the smallest standard error for the European payer and the highest for the monthly Bermudan. Underlying model is the Shifted Gamma with $a = 1.2028$ and $b = 5.9720$.

6. CONCLUSIONS

We showed how a CDS spread simulator based on a firm's value model driven by a single-sided Lévy process can be set up. The spread generator was used to price vanilla and exotic options on single name CDSs by employing Monte Carlo methods. Essentially, after calibrating on a CDS term structure, we simulate the firm's value under the out of the calibration obtained jump dynamics. The paths of the firm's value are then mapped into paths of the corresponding CDS spread. Out of this spread one can calculate straightforwardly the swaption payoff. We have priced in that way, the classical vanilla payers and receivers and have moreover calculated the corresponding (Black's) implied volatilities. It is important to note that the proposed models were capable to generate realistic implied volatility smiles. Next, we have illustrated the strength of the method by pricing several exotic payoff structures of European type. Further, we have used the already documented regression based techniques to price by Monte Carlo methods derivatives of American or Bermudan style under the current jump model setting. From a numerical point of view, the method is based on a double Laplace inversion together with the standard techniques of random number generation and hence the calculations can be performed in terms of seconds on an ordinary computer. Further research goes into the direction of pricing Constant Maturity CDSs and other kinds of derivatives.

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MODELLING AND VALUATION OF DEFAULT DEPENDENCIES IN A TOP-DOWN FRAMEWORK

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Abstract

For this proceeding we summarize the content of our papers Kunisch and Uhrig-Homburg (2007, 2008). In these two papers we provide theoretical and practical insights to a new approach in modelling dependent defaults within a top-down framework. As a result the procedure combines the advantages of reduced form and structural approaches in modelling dependent defaults. Using the smart construction of our default model, valuation formulas for the three major payment blocks are easily derived and applied to credit derivative pricing.
Furthermore, we compute the dependency and the fair spreads of credit derivatives in our framework for given parameter values and compare these results to those ones obtained within basic multi-entity structural and reduced form models.

1. INTRODUCTION AND LITERATURE REVIEW

Recent problems in American subprime mortgage market have sparked off a lively discussion on the pros and cons of credit derivatives. The main focus concerns multi-name contracts such as collateralized debt obligations (CDO's) or CDO-squared contracts (a type of CDO where the underlying portfolio includes CDO tranches). When evaluating these contracts, the crucial point is to capture the dependencies within the portfolio. However, this poses a big challenge to existing pricing models.

Reduced form models introduced by Lando (1998) and Duffie and Singleton (1999) have become a popular tool for pricing credit derivatives like Credit Default Swaps (CDS's) and Collateralized Debt Obligations (CDO's). These models are adapted to some economic state variables so that they can relate credit spread correlations and rating downgrades to the business cycle. However, conditioning on the state variables defaults become independent events within standard doubly stochastic models. In particular, because of their inherent structure, these approaches are unable to explain default clusterings. As a consequence, they produce default correlations that are too small compared to empirical observations[1]. In fact, they would never predict events such as the subprime crisis in fixed income markets and therefore they effectively underprice the risks in credit derivative contracts. To overcome this problem, Jarrow and Yu (2001) allow the intensity to depend on the state of other firms in the economy. Specifically, for some firm i they assume a piecewise constant intensity process with a jump at the default time of another firm j.[2] This approach captures default contagion but loses manageability if the number of firms increases. In particular, the valuation of defaultable claims within such a contagion model becomes quite complicated as the formulas do not only depend on the macroeconomic variables used but also on the current and possible future default statuses of the firms considered. Instead of modelling causal relationships between firms directly, recent approaches put forward by Collin-Dufresne et al. (2003) and Schönbucher (2004) assume that the market dynamically learns from defaults. Building on unobserved frailty variables, new information is revealed through defaults and the market updates its prior distribution.

In Kunisch and Uhrig-Homburg (2007) as well as in Kunisch and Uhrig-Homburg (2008) we propose a new approach to model default dependencies in the context of reduced form models. The main idea of our model is to incorporate contagion effects in addition to the dependence on common state variables in such a way that the model remains to be in a doubly-stochastic setting. This allows us to derive valuation formulas for defaultable claims along the lines of Lando (1998) and therefore simplifies valuation in the presence of contagion effects considerably. Our approach differs by focusing on firm portfolios instead of starting with a description of a default process for each firm individually. By allowing two or more firms to default simultaneously the model captures more realistic levels of default dependencies compared to standard doubly stochastic approaches. A key feature is that the default intensity of a set of firms can be characterized through an economy-wide intensity process and random-thinning using structural firm information. Therefore, both, economy-wide default risk and correlations between firm's asset values induce default dependencies. This is consistent with the idea of markets learning from defaults as well as direct causal relationships between firms. As a result, we obtain a flexible tool for modelling default dependencies that combines the mathematical attractiveness of reduced form models and the economic appeal of structural models.

In the following two sections, Section 2 and Section 3, we give a brief overview of the underlying economy in the model and specify the construction of the top-down approach. This construction is mainly based on the default intensity of sets and derived by random thinning. At this stage structural firm information and relations between firm's asset values are incorporated. After having specified the default model we apply our framework to the valuation of defaultable claims and derive valuation formulas for the three main building blocks in Section 4. In particular, we apply these building blocks to price a credit default swap (CDS) on a single-name and a first-to-default swap on a basket of firms. In Section 5 we show the results from Kunisch and Uhrig-Homburg (2008) in which the impact of a variation of the main dependency drivers on the selected dependence measure as well as on the fair spreads of credit derivatives is illustrated and compared to the selected benchmark models. Section 6 concludes with some final remarks and ideas for further research.

[1]See e.g. Hull and White (2001) and Das et al. (2007).

[2]In Schönbucher and Schubert (2001) jumps at default times are captured through a prespecified copula.

2. A DEFAULT MECHANISM COVERING DEFAULT DEPENDENCIES

2.1. Economic framework

Consider an economy that consists of n firms, whose default status is represented by the vector-valued counting process $N = (N^{(1)}, \ldots, N^{(n)})$ with $N^{(i)}(t) = \sum_{l=1}^{\infty} \mathbf{1}_{\{\tau_l^{(i)} \leq t\}}$. The random variable $\tau_l^{(i)}$ indicates the l^{th} default time of company i. This implies that a firm can default more than once as it is the case after restructuring[3]. Alternatively one could think of replacing any defaulted firm by an otherwise identical non-defaulted firm. In any case, from an economic point of view, this ensures that the universe of firms is kept fixed in the economy instead of reducing the overall number of firms more and more after each default.[4]

Motivated by the work of Davis and Lo (2001) who model a direct default of firm j after the default of firm i with some probability q, we explicitly allow for common jumps in the counting process N. Since this is not in line with the assumption of a Poisson process for the individual events[5], we focus on sets of firms instead of single companies. Defaults of firms that are directly caused by the default of another firm are interpreted as one common event. Therefore it is useful to transform the counting process into an equivalent one based on sets, which is denoted as $\widetilde{N} = (\widetilde{N}^{\{1\}}, \ldots, \widetilde{N}^{\{1,\ldots,n\}})$. This process has now $2^n - 1$ components which represent possible default events. At default only one of these set-based counting processes jumps by one. In this sense these events are called independent.

The uncertainty in this economy is described by the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$, where Ω represents the set of all possible states, $\mathcal{F} = \mathcal{F}_T$ is the σ-algebra on Ω, and $\mathbb{P}$ is the actual probability measure on $\mathcal{F}$. Assuming arbitrage free markets, an equivalent martingale measure $\mathbb{Q}$ exists.[6] Our valuation formulas for defaultable claims in Section 4 rely on this valuation measure.

The model is based on an economy-wide state vector X_t with macroeconomic and possibly firm-specific variables generating the information filtration $\mathcal{F}_t^X = \sigma(X_s : 0 \leq s \leq t)$. Default information concerning the n firms is captured by $\mathcal{H}_t = \bigvee_{i=1}^n \mathcal{H}_t^i$ with $\mathcal{H}_t^i = \sigma(N_s^{(i)} : 0 \leq s \leq t)$. The combined filtration $\mathcal{F}$ is the smallest sigma field that contains $\mathcal{F}^X$ and $\mathcal{H}$, i.e. $\mathcal{F}_t = \mathcal{F}_t^X \vee \mathcal{H}_t$.

Based on this economy, a general valuation framework can be developed in three steps. First, the economy-wide intensity has to be specified. In a second step, we have to split up this economy-wide default risk at the single entity or sub-portfolio level in order to get a representation for the default risk of a single firm. For this purpose we apply random thinning on the economy-wide intensity. Finally, using the conditional independence of the event times of $\widetilde{N}$ with respect to $\mathcal{F}_t^X$, valuation formulas along the lines of Lando (1998) can be derived.

[3]Credit events by the ISDA are: bankruptcy, obligation acceleration, obligation default, failure to pay, repudiation, and restructuring.

[4]This is similar to the case of the iTraxx portfolio in which defaulted CDS will be replaced by non-defaulted ones each six month.

[5]At this point neither the type of process driving N nor a dependence structure among the intensities $\lambda^{(i)}$ belonging to $N^{(i)}$ is assumed.

[6]Under $\mathbb{Q}$ discounted asset prices are martingales with respect to the filtration $(\mathcal{F}_t)_{0\leq t\leq T}$. See Harrison and Pliska (1981) for details.

2.2. Top-down approach

As a starting point of our top-down approach we model the sequence of economy-wide default times $(\tau_j)_{j\in\mathbb{N}}$ as jump times of a doubly stochastic Poisson process with intensity process $\bar{\lambda} = (\bar{\lambda}_t)_{0\leq t\leq T}$. The $\mathcal{F}^X$-adapted[7] process $\bar{\lambda} = (\bar{\lambda}_t)_{0\leq t\leq T}$ describing the economy-wide default intensity that at least one firm defaults corresponds to a jump in one component of $\widetilde{N}$ or equivalently to a jump in at least one component of N.[8] Given a default time τ_j based on the economy-wide intensity process $\bar{\lambda}$, it is now necessary to identify the corresponding set of defaulted firms. In general, there exist $2^n - 1$ possible combinations of firms that could default. These sets form the mark space $\mathbb{X} = \mathcal{P}(\{1,\dots,n\}) \setminus \emptyset$. At each default time τ_j we now draw one element $A_j \in \mathbb{X}$ from this mark space. Together with the default times, they form a marked Poisson process $\Phi = (\tau_j, A_j)_{j\in\mathbb{N}}$. We define $\tau^A = \{\tau_j | A_j = A\}$ as the subsequence of default times where exactly the set A defaults. For each $A \in \mathbb{X}$ we define a counting process $\widetilde{N}^A$ that is zero until the set A is drawn, then at τ_1^A jumps to one and stays there until the next default of this set. By summing over all possible $\widetilde{N}^A$ we get a counting process for the number of default times in the economy: $\widetilde{N}_t^{\Sigma} := \sum_{A\in\mathbb{X}} \widetilde{N}^A$. Consequently, this process obeys the same doubly-stochastic Poisson process that governs the credit event times τ_j in the economy. Furthermore, it holds that $\widetilde{N}_t^{\Sigma} - \int_0^t \bar{\lambda}_s ds$ is a $\mathcal{F}_t^X$-martingale. We can interpret this intensity $\bar{\lambda}$ as the systematic default arrival risk of a given economy. Based on this economy-wide process we next derive a relationship between the intensities of the components of $\widetilde{N}$ and the economy-wide default intensity.
Suppose that for each $A \in \mathbb{X}$ there exists a $\mathcal{F}_t^X$-measurable intensity λ_t^A which is integrable, i.e. $\int_0^t \lambda_s^A ds < \infty, \mathbb{P}-a.s.$, and for which $\widetilde{N}^A - \int_0^t \lambda_s^A ds$ is a martingale. Then, the components $\widetilde{N}^A$ of $\widetilde{N}$ are independent $\mathcal{F}_t^X$-adapted doubly-stochastic Poisson processes[9] with intensities λ_t^A for each $A \in \mathbb{X}$.

In order to derive a relationship between the intensities λ^A and the economy-wide intensity $\bar{\lambda}$, a theorem from Brémaud (1981) is helpful. Since every component $\widetilde{N}^A$ has an intensity λ_t^A, then for all $A \in \mathbb{X}$ it holds[10] that

$$\frac{\lambda_t^A}{\bar{\lambda}_t} = \mathbb{P}\left(A_j = A | \mathcal{F}_{t-}^X, d\widetilde{N}_t^{\Sigma} = 1\right) =: Z_t^A. \tag{1}$$

Note that A_j is the random mark associated to the next default time τ_j. Especially, it holds for each $A \in \mathbb{X}$ and all $j \geq 1$ that $\lambda_{\tau_j}^A / \bar{\lambda}_{\tau_j} = \mathbb{P}\left(A_j = A | \mathcal{F}_{\tau_j -}^X\right)$ on $\{\tau_j < \infty\}$. As $\bar{\lambda}_t$ is the intensity of $\widetilde{N}_t^{\Sigma}$, it also follows that $\bar{\lambda}_t = \sum_{A\in\mathbb{X}} \lambda_t^A$.

Roughly speaking, $\lambda_t^A / \bar{\lambda}_t$ in equation (1) is the probability of having a jump in $\widetilde{N}_t^A$ at time t given $\mathcal{F}_{t-}^X$ and given that $\widetilde{N}_t^{\Sigma}$ has a jump of one[11] at time t.

[7]Instead of assuming an intensity adapted to the underlying state vector only, we could also allow an $\mathcal{F}_t$-adaption. This would mean that the value of the economy-wide intensity can also be affected by individual default events.

[8]Therefore the jumps in $\widetilde{N}$ follow a Poisson process as only one event can happen at default. Note this does not imply an assumption about the underlying point process of N because multiple simultaneous jumps can occur in N.

[9]See Brémaud (1981), Theorem 6, p. 26.

[10]See Brémaud (1981), Theorem 15, p. 34.

[11]The jump-size of one is no restriction since default times τ_j are identified with a unique set. Therefore only the default of one set $A \in \mathbb{X}$ can occur. A simultaneous default of an additional set B in the same short period of time is not possible. The event that both sets default at the same time is already governed by the set $A \cup B = C \in \mathbb{X}$.

This leads to a characterization of λ_t^A:

$$\lambda_t^A = Z_t^A \cdot \bar{\lambda}_t. \tag{2}$$

The probability $\mathbb{P}\left(A_j = A | \mathcal{F}_{t-}^X, d\widetilde{N}_t^{\Sigma} = 1\right) = \mathbb{P}\left(d\widetilde{N}_t^A = 1 | \mathcal{F}_{t-}^X, d\widetilde{N}_t^{\Sigma} = 1\right) = Z_t^A$ defines the random thinning process for the component A. Thus, we can interpret Z_t^A as the conditional probability that the set of firms A defaults at the next default time given $\mathcal{F}_{t-}^X$ and given that there is a jump in $\widetilde{N}$ in the next time increment. Note that the $\mathcal{F}^X$-measurability of the economy-wide intensity is not sufficient to ensure the $\mathcal{F}^X$-measurability of λ^A. In addition, the thinning process must be $\mathcal{F}^X$-measurable. This issue is addressed in the next section where we focus on the question how to express the thinning probabilities Z_t^A using structural firm information.

3. SPECIFICATION OF THE THINNING PROCESS

3.1. The concept of random thining

The concept of random thinning is used to construct the intensity of a single defaulted firm set A from the economy-wide default intensity. Formally, a thinning process for the compensator $\int_0^t \bar{\lambda}_s d\widetilde{N}_s^{\Sigma}$ is a bounded, nonnegative predictable process $Z = (Z^{\{1\}}, \ldots, Z^{\{1,\ldots,n\}})$ for which the Stieltjes integral

$$\int_0^t Z_s^A \bar{\lambda}_s ds$$

defines the indicator process of a totally inaccessible stopping time. If Z is a thinning process, the component Z^A of Z replicates the counting process $\widetilde{N}_t^A = \int_0^t Z_s^A d\widetilde{N}_s^{\Sigma}$. If Z^A is a thinning process for the compensator $\int_0^t \bar{\lambda}_s ds$ that replicates N^A, then the compensator $\int_0^t \lambda_s^A ds$ to $\widetilde{N}_t^A$ is given by[12]

$$\int_0^t Z_s^A \bar{\lambda}_s ds.$$

It can be easily shown, that the thinning process Z^A satisfies the following properties almost surely:

(a) $Z^A \in [0, 1]$ $(A \in \mathbb{X})$,

(b) $\sum_{A \in \mathbb{X}} Z^A = 1$.

These properties can be interpreted in the following way. Firstly, the values of the thinning process are bounded within the interval $[0, 1]$. This property ensures that the intensity λ^A remains positive and that no set carries more default risk than the economy as a whole. Secondly, the entire economy-wide default risk has to be divided across the possible sets. As these properties agree with those of an ordinary probability measure, we refer to Z^A also as thinning probabilities. Usually[13], a third property is to set the Z^A to zero where A defaults. This restriction is not necessary

[12] See Dellacherie and Meyer (1982).

[13] See Giesecke and Goldberg (2007), p. 10.

in our setup since multiple defaults are allowed. Technically, this ensures that the Z^A can be kept $\mathcal{F}^X$-adapted, and therefore do not depend on the default status of the n firms.

The central question is now how to specify this thinning process. As this process only has to admit to properties (a) and (b), there are several degrees of freedom remaining. In what follows we suggest connecting structural firm information, especially the dynamic of the firm's asset values, the structure of liabilities, and the asset correlations to the thinning probabilities. Using a simple structural model like the Merton (1974) model, we can calculate how likely a certain combination of firms ends up in a future state with debt values being higher than asset values (overindebtedness). This fits our idea of firms being closer to overindebtedness being more likely to experience a sudden default event. With these thinning probabilities we can reasonably split up the economy-wide default intensity to the single entity level. Thus, the thinning process Z which in general varies with time and state, partitions the economy-wide default intensity among all possible set-valued events with no default mass being lost.

3.2. Thinning based on structural firm information

To specify the thinning probabilities Z^A $(A \in \mathbb{X})$ based on structural firm information, we introduce the components of a structural model: assume that the value of the total assets of each company follows an $\mathcal{F}^X$-adapted stochastic process $V^{(i)} = (V_t^{(i)})_{t\geq 0}$ $(i = 1, \dots, n)$ and that the asset value processes are correlated with each other. The matrix $\Sigma = (\varrho_{ij})_{i,j=1,\dots,n}$ captures these asset correlations. As $L^{(i)}$ we denote the value of firm i's outstanding liabilities. Based on these specifications the actual likelihood for a set of firms to be overindebted within some given horizon T from now is

$$\widehat{p}^A(t) = \mathbb{P}(V_{t+T}^{(i)} \leq L^{(i)} \; \forall i \in A, V_{t+T}^{(j)} > L^{(j)} \; \forall j \in A^c | \mathcal{F}_{t-}^X),$$

where $A^c = \{1, \dots, n\} \setminus A$ is the complement of the defaulted firm set A. To reach a partition, we standardize $\widehat{p}^A(t)$ with $\sum_{A\in\mathbb{X}} \widehat{p}^A(t)$ which yields to the desired thinning Z_t^A for set A. Obviously, $Z_t^A = \widehat{p}^A(t)/\sum_{A\in\mathbb{X}} \widehat{p}^A(t)$ fulfills properties (a) and (b) by construction. By applying equation (2) we get the desired default intensity λ_t^A of set A. Note that this procedure holds for arbitrary dynamics of $V_t^{(i)}$ and $L^{(i)}$. In order to have a closed-form solution, adequate dynamics have to be specified.

Using a generalized Merton (1974) model, we are able to derive a closed-form solution. The dynamics of the asset value $V^{(i)}$ follow a geometric Brownian motion:

$$dV_t^{(i)} = \mu_i V_t^{(i)} dt + \sigma_i V_t^{(i)} dW_t^{(i)} \quad (i = 1, \dots, n) \tag{3}$$

with constant drift and volatility parameters μ_i and σ_i, respectively. The liabilities are assumed to be constant. This leads to an extended version of the classical Merton framework that allows for correlation among the firm's asset value processes. In the next step, we compute the probability that the firms are overindebted at the fixed horizon T given the current value $V_t^{(i)}$. Because $V_{t+T}^{(i)}$ is log-normally distributed with

$$V_{t+T}^{(i)} = V_t^{(i)} \exp\left((\mu_i - \frac{1}{2}\sigma_i^2)T + \sigma_i W_T^{(i)} \right), \tag{4}$$

it follows

$$\{V_{t+T}^{(i)} < L^{(i)}\} \iff \left\{ U_t^{(i)} < \underbrace{-\frac{\ln(V_t^{(i)}/L^{(i)}) + (\mu_i - \frac{1}{2}\sigma_i^2)T}{\sigma_i\sqrt{T}}}_{d_i} \right\}.$$

Note that the $U_t^{(i)}$ and $U_t^{(j)}$ are standard normal random variables whose correlation is ρ_{ij}. Therefore, the probability $\widehat{p}^A(t) = \Phi_n(-d_1, \ldots, -d_n; \Sigma)$ is the value of a n-dimensional normal distribution with correlation matrix Σ and integration limits $-d_i$. For the companies belonging to the defaulted firm set A we integrate from $-\infty$ to $-d_i$ and for companies that are not in this set A from d_i to ∞. Having calculated $\widehat{p}^A(t)$, Z_t^A results as shown above. So, we obtain a thinning process, that reflects the likelihood that the set A of firms defaults, given at least one sudden default event occurs in the economy.

4. VALUATION FORMULAS FOR DEFAULTABLE CLAIMS

After having introduced and specified our default mechanism, we derive valuation formulas for defaultable claims within the proposed top-down approach in this section. Following Lando (1998) we derive expressions for the three basic building blocks for the payments of defaultable claims and give formulas for single-name credit default swaps and first-to-default swaps[14]. All the expectation are under the risk neutral measure $\mathbb{Q}$. The change from the physical measure to the risk neutral one can be done analogously to Jarrow et al. (2005). Once the economy-wide default intensity $\bar{\lambda}$ is given under the measure $\mathbb{Q}$, we easily obtain the thinned process λ^A under $\mathbb{Q}$ using the same thinning process Z^A as before[15].

Our default model works on sets and not on the default status of the individual companies. At the first sight, this seems to be a problem because defaultable products in the market typically refer to a company and not to a set. Consider the event $\{\tau^{(i)} > T\}$, i.e. the outcome that company i does not default by T. This happens until T if and only if the sets that contain the company i have not suffered a default, i.e. $\tau^{(i)} = \bigwedge_{A\in\mathbb{X}, i\in A}\tau^A$. Since the components of $\widetilde{N}$ are conditionally independent, the intensity of $\tau^{(i)}$ is then simply[16] $\sum_{A\in\mathbb{X}} \lambda_s^A \cdot \mathbf{1}_{\{i\in A\}}$.

Besides default risk a valuation model requires a model of the default-free term structure of interest rates. The term structure is defined by an instantaneous short-rate process $(r_t)_{0\leq t\leq T}$ which is an $\mathcal{F}^X$-adapted stochastic process. The value of a default-free zero-coupon bond at time t with maturity T is:

$$b(r, t; T) = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T r(X_s, s)ds\right) | \mathcal{F}_t^X\right].$$

For ease of notation, we simply write r_s instead of $r(X_s, s)$.

Basically, defaultable claims consists of three major types of payment: promised payments at a

[14]The detailed proofs and derivations can be found in Kunisch and Uhrig-Homburg (2007). Here for the sake of shortness we show only the resulting formulas.

[15]Of course the alternative way consisting of changing the measure of λ^A directly can be done as shown in Jarrow et al. (2005). The results stay the same, due to the multiplicative structure of thinned intensities $Z^A \cdot \bar{\lambda}$.

[16]See Duffie (1998), p. 3.

fixed point in time T, a payment stream up to default or maturity, and payments in the case of default. These payments are now modelled in the following way.

- A payment $Y \in \mathcal{F}_T^X$ at a fixed time T occurs if there had been no default up to this point in time.
- A payment stream such as coupon or swap payment is described by an $\mathcal{F}_t^X$-adapted process C_t. This payment stops when a default occurs or when maturity is reached.
- At the time of default a recovery payment $R_{\tau^{(i)}}$ is an $\mathcal{F}_t^X$-adapted stochastic process. Note that the timing and amount of this payment is random in general. After the expiration of the referenced defaultable claims this payment is set to zero.

For those three blocks we get now the following identities[17]:

$$\mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T r_s ds\right) Y \cdot \mathbf{1}_{\{\tau^{(i)}>T\}}\middle| \mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T\left(r_s + \sum_{A\in\mathbb{X}, i\in A}\lambda_s^A\right)ds\right)Y\middle| \mathcal{F}_t^X\right]$$

$$\mathbb{E}^{\mathbb{Q}}\left[\int_t^T \exp\left(-\int_t^s r_u du\right)C_s \cdot \mathbf{1}_{\{\tau^{(i)}>s\}}ds\middle| \mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[\int_t^T \exp\left(-\int_t^s\left(r_u + \sum_{A\in\mathbb{X}, i\in A}\lambda_u^A\right)du\right)C_s ds\middle| \mathcal{F}_t^X\right]$$

$$\mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^{\tau^{(i)}} r_u du\right)Z_{\tau^{(i)}}\mathbf{1}_{\{\tau^{(i)}\le T\}}\middle| \mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[\int_t^T \sum_{A\in\mathbb{X}, i\in A}\lambda_s^A \exp\left(-\int_t^s\left(r_u + \sum_{A\in\mathbb{X}, i\in A}\lambda_u^A\right)du\right)Z_s ds\middle| \mathcal{F}_t^X\right].$$

With those three blocks most of the existing credit derivatives can be modelled and valuated. Next, we apply these valuation blocks to derive expressions for several defaultable claims. We separate the single-name defaultable claims from the multi-name products. Here, we focus on single-name CDS's and first-to-default swaps as multi-name defaultable claims.

Suppose now that every firm in the economy has a zero-coupon bond with maturity T. The i^{th} single-name CDS only protects against the default of firm i whereas the first-to-default swap refers to a basket containing all the outstanding zero-coupon bonds in the economy. In both cases the protection buyer pays a constant, continuous premium w to the protection seller up to maturity or default. The protection seller compensates the buyer in the case of default for his suffered losses. This payment is specified as $(1-\delta)b(\cdot, T)$, where δ is the recovery rate and $b(\cdot, T)$ the value of an identical default-free zero-coupon bond at the default time. By valuating cash-flow streams of both parties and using the identity these two values, we can solve for the fair premiums w^{cds} and w^{ftd} in each case:

$$w^{cds} = \frac{(1-\delta)\left(b(t,T) - \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T\left(r_u + \sum_{A\in\mathbb{X}, i\in A}\lambda_u^A\right)du\right)\middle| \mathcal{F}_t^X\right]\right)}{\mathbb{E}^{\mathbb{Q}}\left[\int_t^T \exp\left(-\int_t^s\left(r_u + \sum_{A\in\mathbb{X}, i\in A}\lambda_u^A\right)du\right)ds\middle| \mathcal{F}_t^X\right]} \tag{5}$$

$$w^{ftd} = \frac{(1-\delta)\left(b(t,T) - \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T(r_u + \bar{\lambda}_u)du\right)\middle| \mathcal{F}_t^X\right]\right)}{\mathbb{E}^{\mathbb{Q}}\left[\int_t^T \exp\left(-\int_t^s(r_u + \bar{\lambda}_u)du\right)ds\middle| \mathcal{F}_t^X\right]}. \tag{6}$$

[17]For technical details like the existence of the expectations see Kunisch and Uhrig-Homburg (2007).

The single-name CDS's in the market are affected by the dependence structure. Those influences though the thinning can be interpreted as surcharges on the individual spread due to contagion effects. In contrast to a single-name CDS, the first-to-default swap pays the protection payment at first default time that ever occurs in the reference basket, not at the first default time of a specific company i. Since the first default time is the minimum of all jump times of $\widetilde{N}$, it has the intensity $\bar{\lambda}$. Therefore, the fair premium of this multi-name defaultable claim is only related to $\bar{\lambda}$ and not affected by the dependence structure. As $\bar{\lambda} > \sum_{A \in \mathbb{X}, i \in A} \lambda^A$ holds, the absolute height of the premium is always above those for a single-name CDS contract.

5. COMPARISON OF DIFFERENT APPROACHES

In order to provide insights into the dependence structure induced by the proposed top-down approach, we compare the results of the top-down approach to those of several benchmark models. We consider an extended Merton (1974) model with correlated asset value processes and an extension of the Lando (1998) model with correlated default intensities.

We compare the top-down approach via simulation to these two benchmarks in two different ways: first, we analyze the differences between the models with respect to the default correlations and the prices of credit derivatives such as first-to-default swaps and single-name CDS's. Secondly, we investigate which factors drive the dependence structure in the models. To this end, we perform a comparative static analysis focusing on the asset correlation and the intensity level as the most important parameters of the thinning and the economy-wide default process.

5.1. Model specification and simulation setting

To compute our model, we have to specify the components for the thinning and the economy-wide default intensity[18]. We assume that the economy-wide default intensity follows a mean-reverting square-root diffusion process. The dynamics of firm values are according to Section 3 supposed to be geometric Brownian motions now under the risk neutral measure $\mathbb{Q}$. For simplicity of illustration we restrict us to the case of two firms. Furthermore, all three increments of the Brownian motions are assumed to be correlated.

The extended Merton model uses the same asset value processes. Default occurs when the asset value process is below the related value of liabilities at the end of the horizon T. Furthermore, we consider a simple reduced form model with correlated intensities. We assume that each individual intensity $\lambda^{(i)}, i = 1, 2$ follows a mean-reverting square-root process. The individual default probability, joint probability, fair premium for the credit derivatives and default correlation for both benchmark models are computed straight forward[19].

In order to ensure consistency between the economy-wide intensity $\bar{\lambda}$ and the individual intensities of the standard doubly-stochastic model, starting values are chosen such that an aggregation

[18]In order to use the models for valuation of defaultable claims, we specify them under the risk neutral measure $\mathbb{Q}$. Consequently, we also compute the default correlations under this measure.

[19]See Kunisch and Uhrig-Homburg (2008) for details.

of the individual values at time $t = 0$ reproduces the initial value of $\bar{\lambda}$. A similar assumption is made with respect to the long-term means, while we assume that the volatility and the mean-reversion parameter are identical for all three processes. Furthermore, we select the values for the different processes such that the individual default probability of the companies in the extended Merton and the extended reduced form model are equal and the probability for at least one default is the same in all three approaches. In the base case, we select the individual default probability to be 1.2% and the probability of at least one default 2.4%. The detailed setting and the corresponding parameters for the different processes can be found in Kunisch and Uhrig-Homburg (2008).

To evaluate the expressions for the correlations[20] and the prices of credit derivatives we perform a Monte-Carlo simulation. Therefore we discretise the continuous specified stochastic processes by applying an Euler approximation. Then, we simulate the values of these processes up to the horizon T of the simulation[21]. Based on 10,000 simulation runs for a given set of parameters, we compute the default probabilities, the default correlation and the fair spreads for the used credit derivatives.

5.2. Results and interpretation

By varying the time horizon of the top-down approach and the two selected benchmarks, figure 1 (left) illustrates the impact on default correlation for all three models.

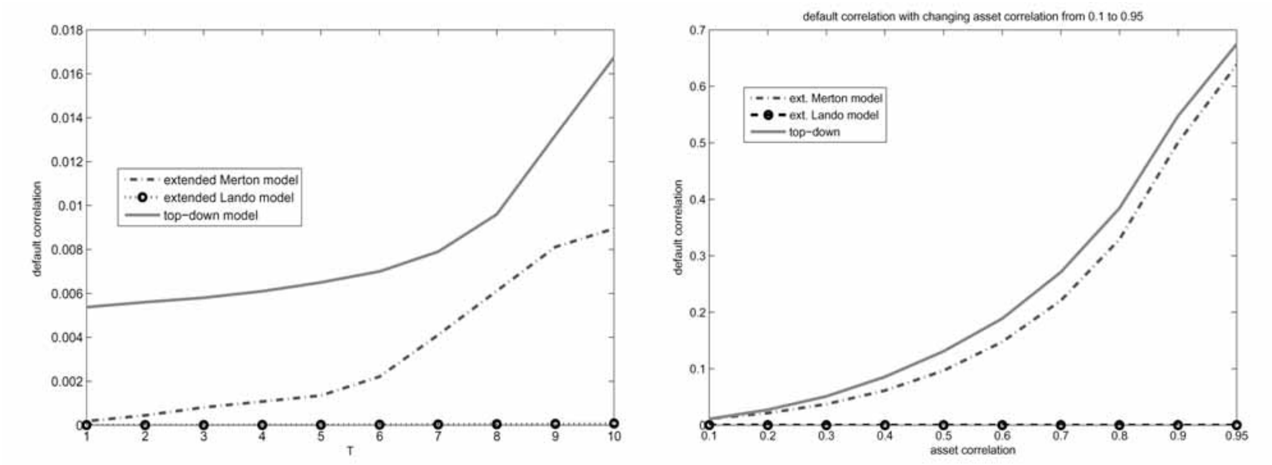

Figure 1: Default correlation in the three approaches with different time horizons (left) and with different asset correlations (right).

As we can see, the extension of the reduced form model is completely unable to produce significantly positive default correlations. Its default correlation is of the order[22] of 10^{-5}. For short maturities, both the structural and the reduced form models produce a negligible correlation. However, with increasing time horizon the default correlation in the structural model increases. This

[20]For the computation of the default correlations in the top-down setting see Kunisch and Uhrig-Homburg (2007).

[21]We choose equidistant time steps with 50 time steps per year. At each time step, we compute the values for the thinning process and calculate the integrated intensity processes.

[22]The value does not change significantly when using other parameters. In particular, an increase in the correlation between the intensities as high as 1 scarcely changes the basic result.

arises from an increasing volatility in the distribution of the asset-value process, i.e. it is more likely that the asset-value process will be below the default barrier. Such a behavior is consistent with the empirical results reported by Lucas (1995). This reported increase of the default correlation with increasing T can also be observed in the top-down approach. In addition, this approach has significantly positive default correlations even for short horizons. So, our parameterization shows that the top-down approach dominates the basic model extensions.
A main driver of the dependence between the firms is the correlation between the asset value process of the two firms (asset correlation). In the extended Merton setting the impact of the asset correlation on the default correlation is well understood. In our top-down approach the asset correlation impacts the default correlation indirectly through the thinning process. To obtain a similar impact in the extended intensity models the only way is through the correlation of the individual intensity processes $\lambda^{(i)}$. Figure 1 (right) illustrates the impact of the microstructure correlation on the default correlation.

Not surprisingly, the increase of the correlation between the two intensities in the extended Lando model has no significant effect on the default correlation. The opposite is true when turning to the extended Merton model and our top-down approach. As we can see these models behave similar when the asset correlation increases. The strong increase in the Merton model is specially due to the fact that we have two symmetric firms and that therefore an increase in the asset correlation has the strongest effects on the event that both will default within the horizon.

In the next step, we vary the starting values of the intensities $\bar{\lambda}_0$ and $\lambda_0^{(i)}$. To make the extended Merton model comparable to these changes in the intensity, we vary the level of the asset value-debt ratio so that the individual default probabilities will be equal. As in the given figures, the change of the intensity level for the doubly-stochastic model has no impact on the default correlation as shown in Figure 2. This is related to the fact that covariance and variance increase by the same magnitude. However, the probability of default increases as expected. In our top-down model and the extended Merton model we can observe an increase in the default correlation whereas the increase in the Merton model is more pronounced. This is due to the fact that default events become more likely and related to this also the intensity for joint events increases which results in a higher default correlation. With the increase of the asset value-debt ratio in the extended Merton model, it is more likely that both companies could be in default at maturity. This results in an increase of the joint probability and leads directly to increasing default correlation.

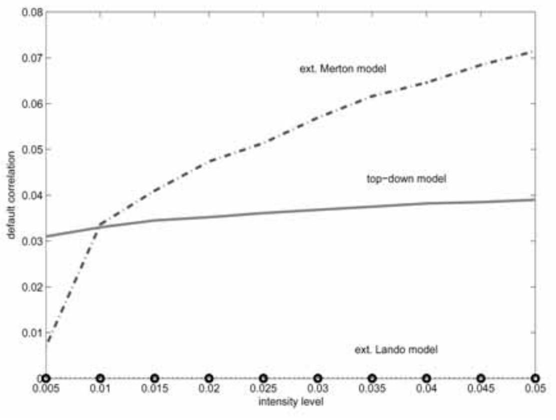

Figure 2: Default correlation with different intensity levels for all three models.

Beside the effect on the dependence measure, a more interesting fact from a practical point of view is how such changes in the default dependencies are affecting fair values of credit derivatives. Here, we use the formulas for CDS and first-to-default swaps from Section 4. We assume that the risk free rate is constant at 3% in order to isolate effects of changes in model parameters. Figure 3 shows the theoretical prices for the two selected credit derivatives under all the three models as we vary the microstructure correlation. By construction the first-to-default swap has the same value for all three approaches because all models are parameterized in such a way that they have the same probability of at least one default. This probability does not depend on the asset correlation. For low values of the asset correlation we only observe a slight difference between the models for value of a single-name CDS. This owes to the fact that all three models have nearly the same individual default probability. The value for a CDS does not change under the extended intensity model and the extended Merton model with varying microstructure correlation because the values used as individual default probabilities are not affected by this correlation. With increasing microstructural correlation the probability that a simultaneous default occurs increases. This increase in the default intensity directly leads to an increasing spread for the contract. We can conclude that only in our model the CDS spread is sensitive to the changes in the asset correlation. The asset correlation in our model controls for the likelihood that the firm defaults as a result of a contagion effect, while the pricing benchmarks neglect this contagion risk.

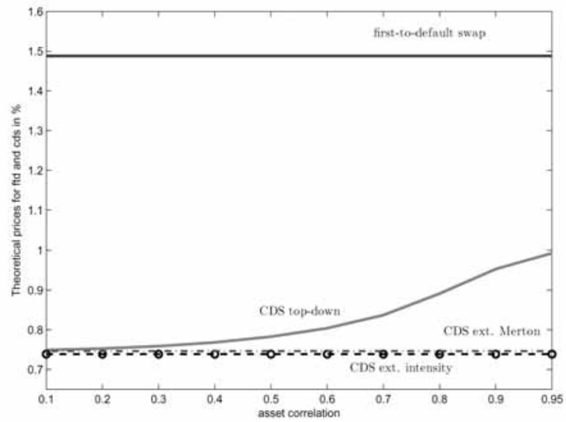

Figure 3: Impact of the asset correlation on theoretical prices of CDS and first-to-default swap.

6. CONCLUSION AND OUTLOOK

In this paper we have introduced a new approach towards modelling default dependencies that is based on a top-down construction. Central element of this construction is the economy-wide default intensity $\bar{\lambda}$. By using random thinning on this process we have been able to derive the intensities for the basic entities. In our model, these entities are not single companies, but sets of companies that can default. This new entity allows for simultaneous defaults of firms and opens a new way in modelling dependencies. To deduce the intensities for the independent default sets we derive the components in the thinning process by using structural elements in a setting similar to

Merton (1974). This procedure incorporates structural information in the setting of a reduced form model while maintaining the advantages of a reduced form model. Furthermore, we can capture two main facts for default dependencies with this top-down construction: common risk factors and contagion effects. The common risk factors drive the stochastic intensity and the correlation between the asset values included in the thinning process. The contagion effects are modelled through the default sets. All sets with more than one firm represent events in which contagion causes multiple defaults. The tendency for simultaneous defaults is mainly driven through the thinning process. Here, the asset correlation is the main driver of this component as illustrated in Section 5. With increasing asset correlation between two companies it becomes more likely that their combination defaults because of the increased value of the associated thinning components.

In fact, the model of Lando (1998) is a special case of our approach. When thinning factors for default sets with two or more elements are set to zero, the contagion factors in the representation of the intensity of company i vanishes and default correlation in that case is only driven by the state variables. Therefore, our approach can be understood as an extension of the standard doubly-stochastic default models allowing for more flexibility in specifying the dependence between firms and therefore producing a dynamic default correlation.

Our setting allows us to construct portfolios with different degrees of diversification and credit derivatives written on these companies. Beside credit derivatives we can also apply our framework to synthetic tranches of credit derivatives indices like, the Dow Jones CDX North American Investment Grade Index or the Dow Jones iTraxx Europe. For this purpose we would have to build a model for the losses that arise in case of a default. This could be easily done in a similar way to Longstaff and Rajan (2008). This may be a promising area of future research on the correlation skew of CDO tranches.

It remains as an open question whether the presented Merton-like approach to derive the thinning process is a promising specification. Further investigation on possible alternatives for the thinning process is needed as well as on other specifications for the economy-wide intensities like e.g. in Errais et al. (2007) and their benefits. It is an empirical question whether this additional complexity is warranted.

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VALUE-AT-RISK AND EXPECTED SHORTFALL FOR RARE EVENTS

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Abstract

We show that the use of correlations for modeling dependencies may lead to counterintuitive behavior of risk measures, such as Value-at-Risk (VaR) and Expected Shortfall (ES), when the risk of very rare events is assessed via Monte-Carlo techniques. The phenomenon is demonstrated for mixture models adapted from credit risk analysis as well as for common Poisson-shock models used in reliability theory.
An obvious implication of this finding pertains to the analysis of operational risk. The alleged incentive suggested by the New Basel Capital Accord (*Basel II*), namely decreasing minimum capital requirements by allowing for less than perfect correlation, may not necessarily be attainable.

1. INTRODUCTION

Since the initiation of the New Basel Capital Accord (*Basel II*) in 1999 when operational risk was introduced to the regulatory landscape, the attention to this risk type has risen substantially. The Committee (Basel Committee on Banking Supervision (2006)) defines operational risk as "risk of loss resulting from inadequate or failed internal processes, people and systems or from external events." The fact that events like bookkeeping errors and terrorist attacks are both contained in this characterization illustrates the broad range of risks, especially when compared to credit or market risk. Taking this heterogeneity of loss events into account, the Basel Committee categorizes losses into seven event types and eight business lines. Banks are supposed to calculate risk measures for each of these $8 \times 7 = 56$ "cells". Examples are "Internal Fraud" in "Trading and Sales" or "Damage to Physical Assets" in "Commercial Banking".

[1]This author presented this paper at the AFMathConf2008 under the name Tina Novotny.

The risk measure specified by the Committee is the *Unexpected Loss* at a confidence level of 99.9%. Generally speaking, this refers to the 99.9% quantile of the loss distribution (possibly reduced by the *Expected Loss*, referring to the mean of the distribution). This quantity is also known as *Value-at-Risk* (VaR), which measures the maximum loss that will not be exceeded with a given confidence level and is widely used in financial institutions since the 1990s.
The total risk capital under the *Advanced Measurement Approaches (AMA)* is obtained by summing over all 56 event-type/business-line VaRs, a strategy implicitly expecting the joint occurrence of all loss types involved or, in other words, perfect positive correlation between all loss processes. The Committee takes this into account by allowing a bank "... to use internally determined correlations [...] provided it can demonstrate to the satisfaction of the national supervisor that its systems for determining correlations are sound, implemented with integrity, and take into account the uncertainty surrounding any such correlation estimates (particularly in periods of stress)." (Basel Committee on Banking Supervision (2006)).
As moving from the highly unrealistic assumption of perfect dependence (summing the Unexpected Losses of all cells) to an approach relying on estimated correlations should lead to a decrease in risk capital, banks have a strong interest in developing and establishing adequate approaches.

This expected decrease in estimated risk capital caused by a lower correlation of loss processes is the focus of our study. We want to find out if a general statement can be made about how risk capital estimates might be altered by such consideration of less than perfect correlation. Secondly, we want to analyze the impact of the concrete model setup on our findings.

In the following, we concentrate on rare event losses, such as natural catastrophes or terrorist attacks, rather than "everyday losses" such as typical bookkeeping errors. Furthermore, we focus on models well-known from credit risk and reliability theory, but with broader parameter ranges than those typically considered. We confine ourselves to analyzing the frequency part of operational losses to check for the impact of dependent occurrences and disregard the severity dimension. Therefore, in our notion, "risk" measures the number of event occurrences rather than monetary units.

Since the work of Artzner et al. (1999) it is well-known that Value-at-Risk (VaR) is not a coherent risk measure. To be precise, it lacks the subadditivity property, which would imply in the context of aggregation of operational risk capital that the joint risk measured for two event-type/business-line cells should not be higher than the sum of the individual risks measured for the two cells. This appears to be a reasonable requirement. Unfortunately, the widely used VaR in general does not fulfil the subadditivity criterion. One recommendation is to calculate the marginal contributions of each business line to the overall risk using conditional expectations and *Expected Shortfall* (ES), i.e, the expected loss given that VaR is exceeded (Glasserman (2005)). However, despite its deficiencies, VaR remains to be the dominant risk measure in practice. Therefore, we consider the two risk measures VaR and ES.

The paper is organized as follows. Section 2 defines latent-variable models and describes the relationship between latent and observed correlation. Mixture models as an alternative representation which offers greater flexibility are presented in Section 3. We introduce a simple common Poisson-shock model in Section 4 and present the results from simulating dependent event occurrences in the aforementioned modeling frameworks in Section 5. Conclusions are presented in Section 6.

2. EVENT OCCURRENCES IN LATENT-VARIABLE MODELS

2.1. Latent-variable models

The idea common to all latent-variable specifications is that there exists a second layer of – possibly observable – latent variables which drive the discrete counting process for the observed loss occurrences. Formally, a latent-variable model (LVM) can be defined as follows, cf. Frey et al. (2005).

Definition 2.1 (Latent-Variable Model) *Let* $X = (X_1, \ldots, X_n)'$, $i = 1, \ldots, n$, *be a random vector and* $D \in \mathbb{R}^{n \times m}$ *a deterministic matrix. Suppose that*

$$S_i = j \quad \Leftrightarrow \quad d_{ij} < X_i < d_{i,j+1} \;, \qquad i \in \{1, \ldots, n\}, j \in \{0, \ldots, m\},$$

where $d_{i0} = -\infty$, $d_{i,m+1} = \infty$. *Then,* (X, D) *is a* latent-variable model *for the state vector* $S = (S_1, \ldots, S_m)'$, *where* X_i *are the latent variables and* d_{ij} *the appertaining thresholds of the latent-variable model.*

For our applications, we introduce a new variable Y_i defined by

$$Y_i = 1 \Leftrightarrow S_i = 0 \qquad \text{and} \qquad Y_i = 0 \Leftrightarrow S_i > 0 \;,$$

to indicate event occurrence, as we only need to distinguish between the two states of "event occurrence" and "non-occurrence". The probability of occurrence for individual/process i is defined by

$$P(Y_i = 1) = P(X_i \leq d_{i1}) = \pi_i \;.$$

In the credit risk literature, $Y_i = 1$ indicates "default" of counterparty i, meaning that obligor i cannot make his payments. In structural credit risk models, the latent variable is interpreted as the obligor's assets; if their value falls below some threshold (the default boundary), the obligor defaults.
This approach can be adapted to suit operational risk settings, but with $Y_i \in \{0, 1, 2, \ldots\}$ the number of loss events rather than the two outcomes "default" or "no default". As a consequence, the Poisson distribution instead of the Bernoulli distribution is appropriate. The Poisson distribution is a natural candidate since it is an approximation for sums of Bernoulli random variables with low success probabilities. This will be realized below in the mixture model representation.

2.2. Latent versus observed correlation

We want to construct a setup in which the probability of the occurrence of an event can depend on events in other processes. Clearly, the probability of a flood damaging equipment will increase when that same event hits a nearby building. Similarly, a system breakdown in one corporate division may propagate to another inducing a failure there. In latent-variable models, this is modeled by allowing for dependence among the latent variables. Thus, dependencies are introduced in an indirect fashion through – typically unobservable – latent variables, X_i, which affect the observed variables, Y_i.

Restricting ourselves to linear dependence, we distinguish between *latent correlation* among the X_i and *observed correlation* among the Y_i, the latter being given by

$$\rho_Y = \frac{\mathrm{Cov}[Y_i, Y_j]}{\sqrt{\mathrm{Var}[Y_i] \cdot \mathrm{Var}[Y_j]}} = \frac{\mathrm{E}[Y_i Y_j] - \pi_i \pi_j}{\sqrt{\pi_i(1-\pi_i)\pi_j(1-\pi_j)}} , \tag{1}$$

where $\mathrm{E}[Y_i Y_j] = P(Y_i = 1, Y_j = 1) = P(X_i \leq d_{i1}, X_j \leq d_{j1})$ denotes the joint cumulative distribution function of the latent variables associated with processes i and j. The observed correlation, ρ_Y, is often called "default correlation" in the credit risk literature, as opposed to (latent) "asset correlation", ρ_X, that refers to the linear dependence between latent variables.

From (1) it follows that observed correlations depend on marginal occurrence probabilities, π_i and π_j, and on latent correlation, ρ_X, the latter entering via $\mathrm{E}[Y_i Y_j]$.

2.3. The distribution of latent variables

Normal variance mixtures are obvious and widely used candidates for the distribution of latent variables. In normal variance mixtures latent variables can be written as

$$X = \mu + \sqrt{W} Z ,$$

where $Z \sim \mathrm{N}_n(0, \Sigma)$, W is a scalar random variable independent of Z and μ is a constant. An event occurs in process i when $X_i \leq d_{i1}$, or

$$Z_i \leq \frac{d_{i1} - \mu}{\sqrt{W}} .$$

The case of multivariate normally distributed latent variables is achieved by setting $\mu = 0$ and $W = 1$. Alternatively, a joint Student-t distribution can be obtained by letting $\nu / W \sim \chi^2_\nu$, where ν is the degrees of freedom parameter of the t distribution and χ^2 denotes the chi-square distribution with ν degrees of freedom. This latter model is often cited in the credit risk literature, e.g. Frey et al. (2001), because it has the appealing feature of treating the KMV and the CreditMetrics model as special cases for which $\nu \to \infty$, but admits lower tail dependence and greater flexibility due to the additional parameter. Other types of latent-variable distributions will be discussed below after having introduced the mixture model representation.

3. EVENT OCCURRENCES IN MIXTURE MODELS

3.1. Mixture models

Mixture models can arise when distributional parameters do not remain constant. For example, it appears to be natural that in times of tectonic plate movements, the probability of an earthquake occurrence rises, that storms are more likely to happen in one season than in others, or that a management change in a global company can affect the probability of fraud. Therefore, in an operational risk context, it seems to be a realistic assumption that the parameters of the assumed distributions might be subject to changes, i.e., be random variables themselves.

A formal definition of a special mixture model in the spirit of Frey et al. (2005) is as follows.

Definition 3.1 (Bernoulli Mixture Model) *Let* $Y = (Y_1, \ldots, Y_n)'$, $i = 1, \ldots, n$, *be a random vector in* $\{0,1\}^n$ *and* $\Psi = (\Psi_1, \ldots, \Psi_p)'$, $p < n$, *be a factor vector. Then,* Y *follows a* Bernoulli mixture model *with factor vector* Ψ *if there exist functions* $p_i : \mathbb{R}^p \to [0,1]$ *such that conditional on* Ψ *the elements of* Y *are independent Bernoulli random variables with* $P(Y_i = 1|\Psi = \psi) = p_i(\psi)$.

It is also possible to define Y as being conditionally Poisson distributed. Then, Y is a count variable rather than a binary variable, and we obtain a *Poisson mixture model*. Both models can be mapped into each other by setting $Y = \mathcal{I}_{\tilde{Y}>0}$ where $\tilde{Y} \sim Poi(\lambda)$. The parameters are related via $p_i = 1 - e^{-\lambda_i}$, a property we will use to simulate from both models in a comparable way.

To keep the setup simple, we examine only exchangeable mixture models, where conditional probabilities of event occurrence are identical, i.e., $p_i(\psi) = p(\psi)$. Defining the new random variable $Q = p(\Psi)$, the observed correlation between indicator variables can then be obtained from

$$\rho_Y = \frac{\pi_2 - \pi^2}{\pi - \pi^2} ,$$

where $\pi = \mathrm{E}[Q]$ and $\pi_k = \mathrm{E}\left[Q^k\right]$.

3.2. Latent-variable models as mixture models

In fact, latent-variable models and Bernoulli mixture models can be viewed as two different representations of the same underlying mechanism. The following lemma is based on Frey and McNeil (2003).

Lemma 3.1 *Let* (X, D) *be a latent-variable model with* n*-dimensional random vector* X. *If* X *has a* p*-dimensional conditional independence structure with conditioning variable* Ψ, *the default indicators* $Y_i = \mathcal{I}_{X_i \le d_{i1}}$ *follow a Bernoulli mixture model with conditional event probabilities* $p_i(\psi) = P(X_i \le d_{i1}|\Psi = \psi)$.

In case of the latent-variable model (X, D) where X is a normal variance mixture and we assume a one-factor structure for Z , we can write

$$\begin{aligned} X &= \mu + \sqrt{W} Z , \\ Z_i &= \sqrt{\rho_X}\, \Psi + \sqrt{1 - \rho_X}\, \varepsilon_i , \end{aligned}$$

where ρ_X is the latent correlation, $\varepsilon_i \sim$ iid $\mathrm{N}(0,1)$, and $\Psi \sim \mathrm{N}(0,1)$ is the only factor and conditioning variable. We thus obtain a conditional independence structure for X, which allows us to proceed using the equivalent mixture model representation. For multivariate normal latent variables with $\mu = 0$ and $W = 1$, the observed conditional default probability is

$$p(\psi) = P(X_i \le d_{i1}|\Psi = \psi) = \Phi\left(\frac{\Phi^{-1}(\pi) - \sqrt{\rho_X}\,\psi}{\sqrt{1 - \rho_X}}\right) .$$

For a multivariate Student-t distribution the analogous result is

$$p(\psi) = P(X_i \leq d_{i1}|\Psi = \psi) = \Phi\left(\frac{t_\nu^{-1}(\pi)W^{-1/2} - \sqrt{\rho_X}\,\psi}{\sqrt{1-\rho_X}}\right) .$$

We see that we can easily map the latent-variable models of Section 2.3 into the mixture model setup; at the same time, simulation is much easier, because we do not have to draw from the multivariate normal or multivariate Student-t probability density function.

3.3. The mixing distribution

Within the mixture model framework, one can easily allow for different distributional assumptions with respect to latent variables. In our analyses, we consider several examples which are often suggested in risk-management and actuarial applications. In each case the model was calibrated to the multivariate normal latent-variable model, to assess to what extent the choice of mixing distribution affects the number of event occurrences, with the multivariate normal model serving as benchmark.

3.3.1. Beta Mixing Distribution

In case of a Beta mixing distribution we assume a mixing variable $Q = p(\Psi) \sim Beta(a, b)$. As the moments of a Beta distribution can be directly calculated from the distributional parameters, a and b, we can easily derive unconditional occurrence probabilities from

$$\pi_k = \frac{\beta(a+k,b)}{\beta(a,b)} = \prod_{j=0}^{k-1} \frac{a+j}{a+b+j} ,$$

from which we obtain the observed correlation

$$\rho_Y = \frac{1}{a+b+1} .$$

3.3.2. Probit Model

We assume a standard normally distributed factor $\Psi \sim N(0,1)$. Conditional event probabilities have to be determined using the fact that $Q = \Phi(\mu + \sigma\Psi)$. Marginal occurrence probabilities are not as easily obtained as in the Beta case, since it involves the integration

$$\pi_k = \mathrm{E}\left[Q^k\right] = \int_{-\infty}^{\infty} (\Phi(\mu + \sigma z))^k \phi(z) dz ,$$

making simulations of event occurrences rather complicated. Matters become much easier when recalling that the Probit model is equivalent to a latent-variable model with multivariate normally distributed latent variables. Hence, this model is already covered by the benchmark model described in Section 3.2.

3.3.3. Latent Variables with Clayton Copula

The Clayton Copula is a subtype of an Archimedean Copula

$$C(u_1, \dots, u_d) = \phi^{-1}(\phi(u_1) + \dots + \phi(u_d))$$

with generator $\phi(t) = t^{-\theta} - 1$ being the inverse of the Laplace transform of cumulative distribution function G on $\mathbb{R}$.
Using $\Psi \sim$ Ga$(1/\theta, 1)$, conditional occurrence probabilities can be calculated from $Q = p(\Psi)$ with

$$Q = p(\psi) = P(U_i \leq \pi | \Psi = \psi) = \exp(-\psi\phi(\pi)) \ ,$$

where $U_i \sim$ Unif$(0, 1)$. The bivariate occurrence probability is

$$\pi_2 = \phi^{-1}(\phi(\pi) + \phi(\pi)) = (2\pi^{-\theta} - 1)^{-1/\theta} \ .$$

4. EVENT OCCURRENCES IN COMMON POISSON-SHOCK MODELS

4.1. A simple common Poisson-shock model

Adapting the frameworks of Powojowski et al. (2002) and Lindskog and McNeil (2003), one can assume the presence of both idiosyncratic and common Poisson processes. Altogether we assume $m = n + n_c$ underlying processes. The number of loss events for the observed loss process i can be written as

$$Y_i = \sum_{j=1}^{m} \delta_{ij} M_j \quad i = 1, \dots, n, \ j = 1, \dots, m,$$

where δ_{ij} is an indicator variable which is equal to one if underlying process j can lead to loss events of observed process i, and M_j represents the number of occurrences of underlying process j with intensity λ_j. Among the m underlying processes there are n_c common ones, which affect more than one observed process and are characterized by equal intensities, λ_c. The remaining n underlying processes with intensities λ_i^* are idiosyncratic in the sense that they only affect the observed process i.
The correlation between two observed loss event processes, k and l, can be written as

$$\rho_{kl} = \frac{\sum_{j=1}^{m} \delta_{jk}\lambda_j\delta_{jl}}{\sqrt{\sum_{j=1}^{m} \delta_{jk}\lambda_j \sum_{j=1}^{m} \delta_{jl}\lambda_j}} \ . \qquad (2)$$

We use a simplified setup comparable to the exchangeable mixture model where idiosyncratic intensities $\lambda_i^* = \lambda^*$ are identical as well and where all n_c common processes cause events in all n observed loss processes. Equation (2) can then be written as

$$\rho_{kl} = \frac{n_c\lambda_c}{\lambda^* + n_c\lambda_c} \ .$$

For a given $\lambda = \lambda^* + n_c\lambda_c$ and observed correlation $\rho_{kl} = \rho$, we can calculate idiosyncratic and common parts from

$$\begin{aligned} \lambda^* &= \lambda(1-\rho) , \\ n_c\lambda_c &= \lambda - \lambda^* . \end{aligned}$$

5. SIMULATION RESULTS

For each of the models discussed above, we simulated event occurrences and estimated risk capital for different levels of latent correlation, ρ_X. In doing this, we used the multivariate normal latent-variable model as benchmark model to which we calibrated the other models.
Throughout the simulations, we assumed $n = 1\,000$ loss processes, to match with the studies in Frey et al. (2001) and Frey and McNeil (2001). For the mixture models, we simulated a factor realization ψ and calculated conditional occurrence probabilities $p(\psi)$ which were then used to conduct n Bernoulli or Poisson trials. After summing up the number of event occurrences, we repeated 100 000 times and calculated VaR and ES of the resulting empirical distribution. For the common Poisson-shock model, we assumed $n_c = 1$ and calculated λ, λ^* and λ_j^c from π and ρ_Y. These quantities were then used to conduct Poisson trials and proceed further as in the mixture model setup.

For all models and low occurrence probabilities ($\pi \leq 0.01$), we observe a counterintuitive behavior of VaR: it decreases for increasing correlations, this effect being the more pronounced the lower the confidence level. An illustration of this phenomenon is given in Figure 1, which plots the logarithm of the 99% VaR depending on the level of latent correlation and occurrence probability π. While for $\pi = 0.01$, VaR behaves as intuitively expected, i.e., increasing in ρ_X over the entire range of latent correlations, it clearly declines for lower levels of latent correlation beyond a certain threshold of ρ_X, with lower thresholds for lower values of π. This effect is more pronounced as the tails of the distribution of latent variables become fatter, as is shown in Figure 2. For $\nu = 100$, we observe an increase in VaR up to a latent correlation of $\rho_X \approx 0.5$ and a decrease for higher levels; the lower ν, the broader the range of ρ_X for which this peculiar behavior occurs. For $\nu = 4$, VaR decreases over the entire range of latent correlations. The results for Poisson mixture models are qualitatively the same, as was to be expected from the low level of occurrence probabilities involved. Also, scenarios for the common Poisson-shock model setup can be established which lead to this counterintuitive behavior.
For ES, using 100 000 replications leads to ambiguous results. For very low occurrence probability levels ($\pi \leq 0.00001$), decreases in ES can be observed for increasing ρ_X. But in contrast to the risk capital estimates based on VaR, this effect vanishes when the number of replications increases to up to 10 million. Figure 3 illustrates that ES behaves as intuitively expected, i.e., it rises over the entire range as ρ_X grows. Therefore, the counterintuitive behavior has to be taken into account when designing the Monte-Carlo simulation. Otherwise, simulated ES figures may seem to decrease as ρ_X rises – just as is in the VaR case.

The explanation of this effect is illustrated in Figure 4. It shows 10 000 draws from a bivariate

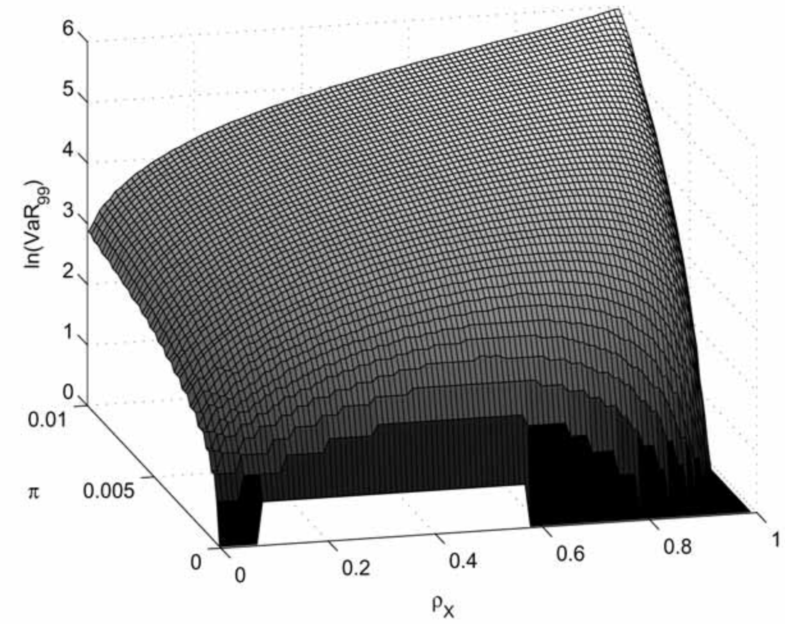


Figure 1: Simulated 99%-VaR (in logs) in the multivariate normal LVM, $\pi \in [0.0001, 0.01]$

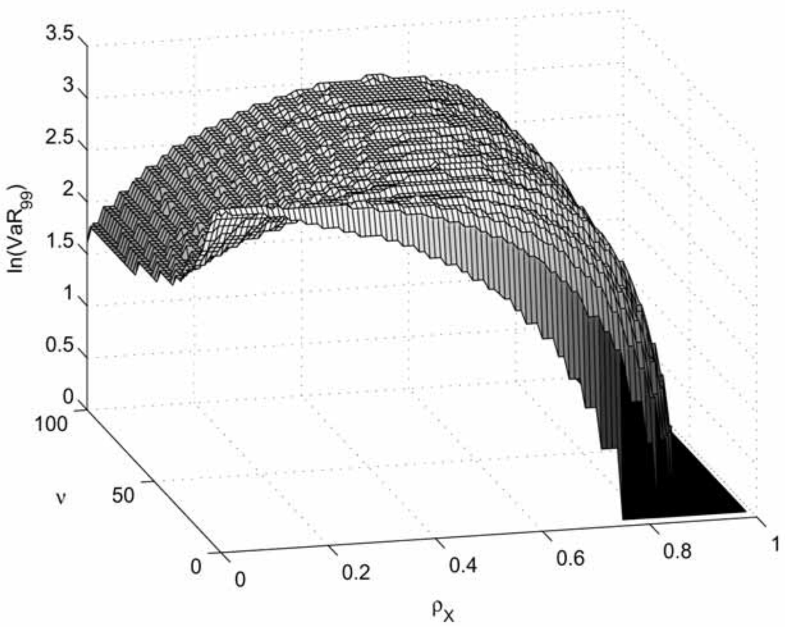


Figure 2: Simulated 99%-VaR (in logs) in the multivariate Student-t LVM, $\pi = 0.001$, $\nu \in [4, 100]$

normal distribution for two different correlation assumptions. The solid line represents the thresholds implied by an occurrence probability of $\pi = 0.01$. In the left plot, where the latent correlation is $\rho_X = 0.1$, this threshold leads to 4 joint "occurrences" (in the southwestern quadrant) and 9,798 joint "non-occurrences". In the right plot with a higher correlation of $\rho_X = 0.9$, the concentration on extremes leads to 94 joint "occurrences" and 9,854 joint "non-occurrences". As it turns out,

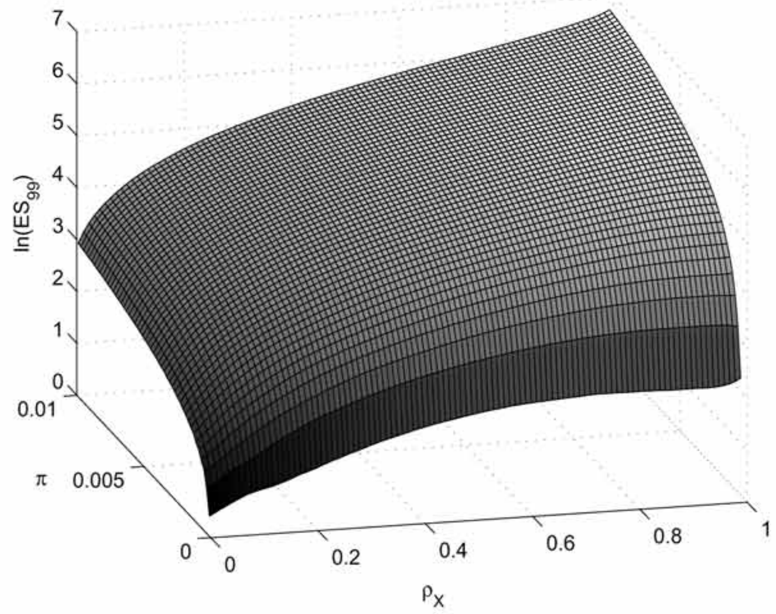
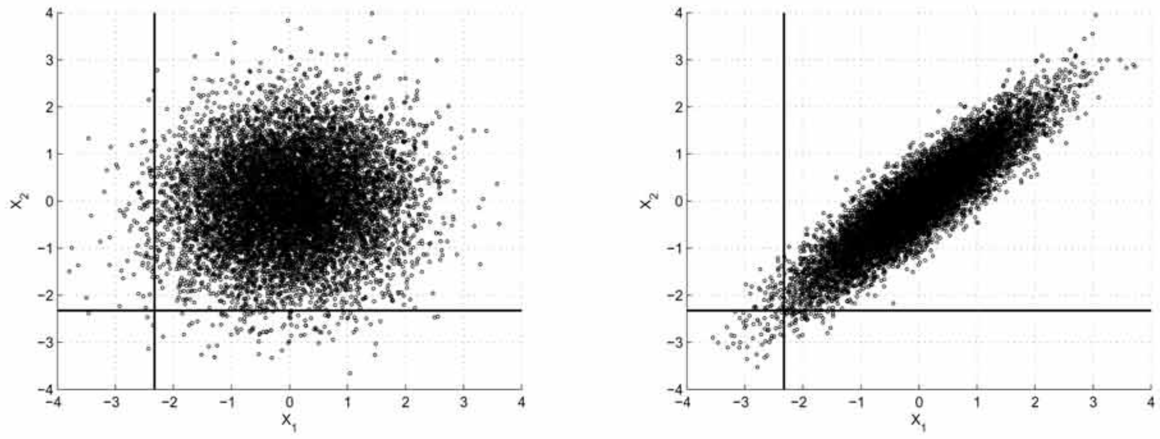


Figure 3: Simulated 99%-ES (in logs) in the multivariate normal LVM, $\pi \in [0.0001, 0.01]$

high correlation not only leads to more events, but also to more joint “non-events”. It is this phenomenon which moves Value-at-Risk towards zero as correlation levels rise.



(i) $\rho_X = 0.1$ (ii) $\rho_X = 0.9$

Figure 4: Scatterplots for a bivariate normal distribution

6. CONCLUSION

Introducing less than perfect dependencies should lead to a more realistic description of loss event occurrences. Our results show that it is very important to assess the impact of correlations within the chosen modeling framework. Be it mixture models, common Poisson-shock models or a different setup, in the case of rare events, simulated values for risk measures, such as Value-at-Risk and Expected Shortfall, can decrease as the level of correlation increases. The parameter ranges for which this phenomenon occurs may not be so relevant for credit risk applications, but may arise in operational risk applications where several business lines at close locations could, for example, be affected by some catastrophic event.
While this effect can be eliminated in the case of Expected Shortfall by an appropriate design of the Monte-Carlo setup, this is unfortunately not so for the widely used Value-at-Risk which systematically declines above certain levels of latent correlations. The extent to which this arises depends on the observed occurrence probabilities, the confidence level and the fat-tailedness of the distribution of the latent variables.
If the clustering of realizations at zero ("joint non-occurrences") that causes this behavior is a misleading feature of the model which contradicts the true risk-generation mechanisms, risk capital can severely be underestimated, and other dependence concepts should be considered for calculating risk capital.

A practical implication of our analysis is that the inclusion of non-perfect correlations in models used for assessing minimum capital requirements for operational risk may, in fact, lead to an increase of the assessed amount.

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OVERVIEW: (LOCALLY) RISK-MINIMIZING HEDGING STRATEGIES

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Abstract

The goal of this article is to give an overview of the theory concerning (locally) risk-minimizing hedging strategies. We also discuss the most important applications and extensions of the locally risk-minimizing hedging strategy. Finally, we show how the multidimensional locally risk-minimizing hedging strategy is defined and we explain why it is not straightforward to find the locally risk-minimizing hedging strategy in case the underlying risky asset follows a geometric Lévy process.

1. INTRODUCTION

The difference between the two commonly used quadratic hedging methods, namely mean-variance hedging and locally risk-minimizing hedging, has already been extensively described and investigated. We call a hedging strategy quadratic if the strategy minimizes the hedging error in mean square sense. One of the obvious drawbacks of quadratic hedging is that losses and gains are treated in the same way.

In the mean-variance hedging theory the goal is to minimize the difference between the claim at time T and the portfolio at that time, using a self-financing strategy. In the risk-minimizing hedging strategy, the goal is to minimize the variance of the cost process at any time t subject to the condition that the value of the portfolio at time T equals H. In the latter case, it is only possible to find a self-financing portfolio when the claim is attainable.

When reading papers concerning the locally risk-minimizing hedging strategy, we noticed some confusion about the notion of this strategy, especially in the case of discontinuous price processes for the risky asset.

Pham (2000) and Schweizer (2001) have already provided a rather theoretical overview of these two quadratic hedging methods, although without warning for the pitfalls when dealing with discontinuous risky assets. In case of discontinuity, orthogonality is not preserved by the change of measure to the minimal martingale measure, and hence there is no longer equivalence between the

Galtchouk-Kunita-Watanabe decomposition under the minimal martingale measure and the needed Föllmer-Schweizer decomposition under the original measure.
Section 2 therefore presents the basic results without proof for the hedging in continuous time, on which the theory concerning locally risk-minimizing hedging strategies is based. We also believe that it would be better to use the notion "Föllmer-Schweizer measure" only for a minimal martingale measure preserving orthogonality, since mistakes are made by overlooking the continuity assumption on the underlying risky asset in the setting of Föllmer and Schweizer (1991).
In 2000 Pham published an overview of the literature on locally risk-minimizing hedging strategies, but since then many new papers on this topic has appeared. For this reason, section 3 contains an addition to the overview in Pham (2000).
In section 4, we briefly describe how the theory of locally risk-minimizing hedging strategies is extended to the multidimensional case. In the end, we show why the equivalence between the Galtchouk-Kunita-Watanabe decomposition under the minimal martingale measure and the Föllmer-Schweizer decomposition under the original measure does no longer hold if the underlying risky asset follows a geometric Lévy process.

2. (LOCAL) RISK MINIMIZATION

2.1. Setting

We introduce a complete probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, with the filtration $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$ containing the information from the risky asset available up to time t. The one-dimensional stochastic process $X = (X_t)_{0\leq t\leq T}$ denotes the evolution of the price process of the discounted risky asset. This process is adapted and has càdlàg paths. The process of the riskless asset is denoted by $B = (B_t)_{0\leq t\leq T}$.
Furthermore, we make the assumption that there exists at least one change of measure to a martingale measure. This is related to some kind of no-arbitrage assumption, for more explanation we refer to Delbaen and Schachermayer (1994).

2.2. Hedging in complete markets

Harrison and Kreps (1979) described how the hedging strategy for redundant contingent claims can be found. Redundant means that the claim can be written as a sum of an initial cost, denoted by C_0, and a stochastic integral of the process of the discounted risky asset. Hence, the risk of the claim can be reduced to zero. Assume we want to hedge the claim H at time T. We perform a change of measure from the original measure P to the unique equivalent martingale measure P^*. Due to the completeness of the market, we know that every claim H can be decomposed as follows:

$$H = H_0 + \int_0^T \xi_u^* dX_u.$$

Then, the contingent claim can be reproduced at time T with the initial investment H_0 and the following strategy at time t:

$$(\xi_t^*, H_0 + \int_0^t \xi_u^* dX_u - \xi_t^* X_t).$$

2.3. Risk minimization

The goal of Föllmer and Sondermann (1986) was to extend the hedging theory for redundant claims to contingent claims which are non-redundant. They searched for admissible strategies which minimize risk in sequential sense and were able to prove that there exists a unique solution to this problem. In the theory of risk minimization, Föllmer and Sondermann assumed that the discounted risky asset is a square-integrable martingale under the original measure P. This means that $E[X_t^2] < \infty$ and $E[X_T|\mathcal{F}_t] = X_t$, $0 \leq t \leq T$. A trading strategy is of the form $\varphi = (\xi, \eta)$, with $\xi = (\xi_t)_{0 \leq t \leq T}$ the amount invested in the risky asset and with $\eta = (\eta_t)_{0 \leq t \leq T}$ the amount invested in the riskless asset. The value of the discounted portfolio at time t is then given by $V_t = \xi_t X_t + \eta_t$.

Definition 2.1 (Hedging strategy in case X is a local martingale) *A couple $\varphi = (\xi, \eta)$ is called a strategy if*

- *ξ is a predictable process,*
- *$\xi \in L^2(X)$, with $L^2(X)$ the space of all $\mathbb{R}$-valued predictable processes ξ such that*

$$||\xi||_{L^2(X)} := (E[\int_0^T \xi_u^2 d[X, X]_u])^{1/2} < \infty,$$

- *η is adapted,*
- *$V = \xi X + \eta$ has right continuous paths and $E[V_t^2] < \infty$ for every $t \in [0, T]$ (i.e. $V_t \in L^2(P)$ for every $t \in [0, T]$).*

The cost process is the difference between the value of the portfolio at time t and the gains/losses made from trading in the financial market up to time t:

$$C_t = V_t - \int_0^t \xi_u dX_u. \tag{1}$$

This process is called self-financing if it has constant paths and it is called mean-self-financing if it is a square-integrable martingale.
We want to hedge a contingent claim $H \in L^2(P)$ due at time T. By searching for a hedging strategy for which the discounted portfolio has terminal value H, we will find a H-admissible strategy. Due to the martingale property of the risky asset process, it is possible to show that the expected value of the terminal cost does not depend on the choice of the strategy: $E[C_T] = E[H]$. At any time t we have to minimize the remaining cost $C_T - C_t$. We measure this risk by

$$R_t^{\varphi} = E[(C_T - C_t)^2|\mathcal{F}_t].$$

A strategy is then called risk-minimizing if $R_t^{\varphi} \leq R_t^{\hat{\varphi}}$ P-almost surely for every admissible continuation $\hat{\varphi}$ of φ at time t. An admissible continuation of the strategy φ is a strategy which coincides with φ for all times smaller than t and which also has terminal value H.
Furthermore, it can be proved that an admissible risk-minimizing strategy is mean-self-financing. Föllmer and Sondermann showed that the solution can be found using the Galtchouk-Kunita-Watanabe decomposition of the contingent claim H:

$$E[H|\mathcal{F}_t] = E[H] + \int_0^t \xi_u^* dX_u + N_t^*,$$

with $\xi^* \in L^2(X)$, N_t^* a square-integrable martingale, $E[N^*] = 0$ and orthogonal to the space $\{\int_0^t \delta_s dX_s \mid \delta \in L^2(X)\}$.
The unique admissible strategy which is risk-minimizing is then given by $\varphi_t^* = (\xi_t^*, E[H|\mathcal{F}_t] - \xi_t^* X_t)$ at time t and the remaining risk equals $E[(N_T^* - N_t^*)^2|\mathcal{F}_t]$.
We remark that in this case the number of risky assets invested using the mean-variance hedging strategy coincides with the number invested using the risk-minimizing hedging strategy. The amount invested in the riskless asset is different.

2.4. Local risk minimization

The foundation for the locally risk-minimizing hedging strategy is described in Schweizer (1990), where the equivalence between the orthogonality of martingales and the risk-minimality under small perturbations is proved. In Schweizer (1993a) the concept of locally risk-minimizing hedging strategies is introduced to be able to hedge claims when the underlying risky asset X is only a semimartingale under the original measure. The types of semimartingales for which the locally risk-minimizing hedging strategy is described have to be of the following form:

$$X = X_0 + Z + A, \tag{2}$$

with Z a square-integrable martingale for which $Z_0 = 0$, and with A a predictable process of finite variation $|A|$ (i.e. $\sup_\tau \sum_{i=1}^{N(\tau)} |A_{t_i} - A_{t_{i-1}}| < \infty$ for every partition τ of $[0, T]$).
One also needs the following assumptions:

(A1) For P-almost all ω, the measure on $[0, T]$ induced by $\langle Z \rangle(\omega)$ has the whole interval $[0, T]$ as its support. This means $\langle Z \rangle$ should be P-almost surely strictly increasing on the whole interval $[0, T]$.

(A2) A is continuous.

(A3) A is absolutely continuous with respect to $\langle Z \rangle$ with a density α satisfying

$$E_Z[|\alpha| \log^+ |\alpha|] < \infty.$$

A sufficient condition is that $E[\langle \int \alpha dZ \rangle] < \infty$.

The definition for the trading strategy has to be adjusted in this case:

Definition 2.2 (Hedging strategy in case of a semimartingale X) *A couple $\varphi = (\xi, \eta)$ is called a strategy if*

- *ξ is a predictable process,*
- *$\xi \in L^2(Z)$ and $\int_0^T |\xi_u dA_u| \in L^2(P)$. This ensures that $\int_0^t \xi_u dX_u$, $0 \le t \le T$ is a semimartingale of class $\mathcal{S}^2$ meaning that*

$$E[\int_0^T \xi_u^2 d\langle Z, Z\rangle_u + (\int_0^T |\xi_u dA_u|)^2] < \infty.$$

The space formed by all the processes ξ satisfying this condition is given by Θ_S.

- *η is adapted,*
- *$V = \xi X + \eta$ has right continuous paths and $E[V_t^2] < \infty$ for every $t \in [0, T]$ (i.e. $V_t(\varphi) \in L^2(P)$ for every $t \in [0, T]$).*

In order to define the notion of locally risk-minimizing hedging strategies, we first explain what is meant by a small perturbation:

Definition 2.3 (Small perturbation) *A trading strategy $\Delta = (\delta, \varepsilon)$ is called a small perturbation if it satisfies the following conditions:*

- *δ is bounded,*
- *$\int_0^T |\delta_u dA_u|$ is bounded,*
- *$\delta_T = \varepsilon_T = 0$.*

For any subinterval $(s, t]$ of $[0, T]$, we define the small perturbation

$$\Delta|_{(s,t]} := (\delta \mathbb{1}_{(s,t]}, \varepsilon \mathbb{1}_{[s,t)}).$$

Next we define partitions $\tau = (t_i)_{0 \le i \le N}$ of the interval $[0, T]$. A partition of $[0, T]$ is a finite set $\tau = \{t_0, t_1, \ldots, t_k\}$ of times with $0 = t_0 < t_1 < \ldots < t_k = T$ and the mesh size of τ is $|\tau| := \max_{t_i, t_{i+1} \in \tau} (t_{i+1} - t_i)$. A sequence $(\tau_n)_{n \in \mathbb{N}}$ is called increasing if $\tau_n \subseteq \tau_{n+1}$ for all n and it tends to the identity if $\lim_{n \to \infty} |\tau_n| = 0$.

Definition 2.4 (Locally risk-minimizing) *For a trading strategy φ, a small perturbation Δ and a partition τ of $[0, T]$, the risk quotient $r^\tau(\varphi, \Delta)$ is defined as follows:*

$$r^\tau(\varphi, \Delta) := \sum_{t_i, t_{i+1} \in \tau} \frac{R_{t_i}(\varphi + \Delta|_{(t_i, t_{i+1}]}) - R_{t_i}(\varphi)}{E[\langle Z\rangle_{t_{i+1}} - \langle Z\rangle_{t_i} | \mathcal{F}_{t_i}]} \mathbb{1}_{(t_i, t_{i+1}]}.$$

A trading strategy φ is called locally risk-minimizing if $\liminf_{n \to \infty} r^{\tau_n}(\varphi, \Delta) \ge 0$ P_Z-a.e. on $\Omega \times [0, T]$ for every small perturbation Δ and every increasing sequence (τ_n) of partitions of $[0, T]$ tending to the identity.

Lemma 2.1 *Assume that the semimartingale X, with the decomposition described in (2), satisfies $(A1)$. If a trading strategy is locally risk-minimizing, then it is also mean-self-financing.*

Proposition 2.2 *Assume that the semimartingale X, with the decomposition described in (2), satisfies all conditions (A1)-(A3). Let the contingent claim H belong to $L^2(P)$ and let φ be a H-admissible trading strategy. Then φ is a locally risk-minimizing strategy if and only if φ is mean-self-financing and the martingale $C(\varphi)$, (1), is orthogonal to the martingale part Z of the semimartingale X.*

Definition 2.5 (Pseudo locally risk-minimizing hedging strategy) *A strategy φ is called pseudo locally risk-minimizing or, equivalently, pseudo optimal risk-minimizing if the associated cost process $C(\varphi)$, (1), is a martingale under P and orthogonal to Z in (2).*

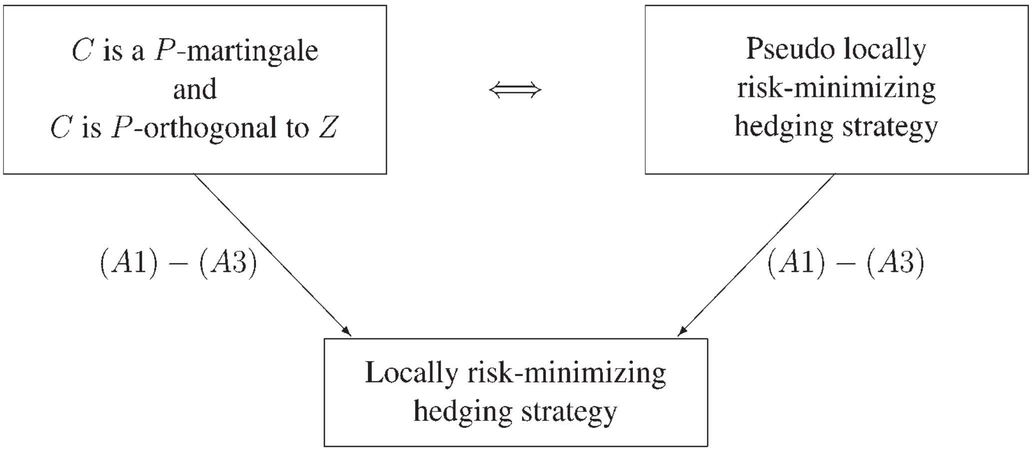

Figure 1: Equivalent conditions to determine the locally risk-minimizing hedging strategy under the original measure P.

Definition 2.6 (Föllmer-Schweizer decomposition)

$$H = H_0 + \int_0^T \xi_u^H dX_u + L_T^H \quad P\text{-a.s.} \tag{3}$$

is the Föllmer-Schweizer decomposition of the contingent claim H if $\xi^H \in \Theta_S$ and if L^H is a square-integrable P-martingale orthogonal to Z (2), *with $L_0^H = 0$.*

Having the Föllmer-Schweizer decomposition, we can directly give the pseudo locally risk-minimizing hedging strategy φ:

$$\varphi_t = (\xi_t^H, H_0 + \int_0^t \xi_u^H dX_u + L_t^H - \xi_t^H X_t).$$

Using proposition 2.2, we know that this pseudo locally risk-minimizing hedging strategy is the locally risk-minimizing strategy if the assumptions (A1)-(A3) are satisfied (see also Figure 1). In some cases, we can easily determine the Föllmer-Schweizer decomposition by performing a change of measure, as we will show here along the lines of Föllmer and Schweizer (1991). They assume that the semimartingale X is continuous and in this case the conditions (A1)-(A3) are trivially satisfied.

Definition 2.7 (Minimal martingale measure) *A martingale measure $\hat{P}$, equivalent with the original measure P, will be called minimal if*

$$\hat{P} = P \quad on \quad \mathcal{F}_0, \tag{4}$$

and if any square-integrable P-martingale which is orthogonal to the martingale part Z of the semimartingale X, as in (2), under P remains a martingale under $\hat{P}$.

Given the continuity of X it is easy to prove that the minimal martingale measure preserves orthogonality. This means that any square-integrable P-martingale orthogonal to Z is also orthogonal to X under $\hat{P}$.
We emphasize that the preservation of orthogonality is not included in the definition of the minimal martingale measure, but that it is a consequence in some special cases.
In those cases, we will call the measure the "Föllmer-Schweizer" measure.

Definition 2.8 (Föllmer-Schweizer measure) *The Föllmer-Schweizer measure is a minimal martingale measure for which orthogonality is preserved by a change of measure from the original measure to the minimal martingale measure.*

Combining the previous results, we have the following proposition:

Proposition 2.3 *If X is continuous, the locally risk-minimizing strategy is determined by the Galtchouk-Kunita-Watanabe decomposition under the minimal martingale measure.*

Proof. Föllmer and Schweizer (1991) proved that the minimal martingale measure preserves orthogonality if X is continuous. In this case the Galtchouk-Kunita-Watanabe decomposition under the minimal martingale measure directly implies the Föllmer-Schweizer decomposition under the original measure. This already gives us the pseudo locally risk-minimizing hedging strategy. Since by the continuity of X, (A1)-(A3) are satisfied, we know from proposition 2.2 that this strategy is locally risk-minimizing under the original measure. ■

As a result of proposition 2.3, we can determine the locally risk-minimizing hedging strategy in case of a continuous risky asset following the scheme in Figure 2.
In case the semimartingale is discontinuous, there is no easy way to find the Föllmer-Schweizer decomposition and hence to determine the locally risk-minimizing hedging strategy. We refer to section 3 for a possible solution to this problem.

3. APPLICATIONS IN LITERATURE

This section presents the most important applications of the locally risk-minimizing hedging strategy. First of all, it is important to pay attention to the different notions of locally risk-minimizing hedging strategy in discrete and continuous time. In discrete time, the concept of locally risk-minimization is often used for the strategy which minimizes the difference between the cost process in every subsequent time interval, e.g. Coleman et al. (2006). This definition allows us to

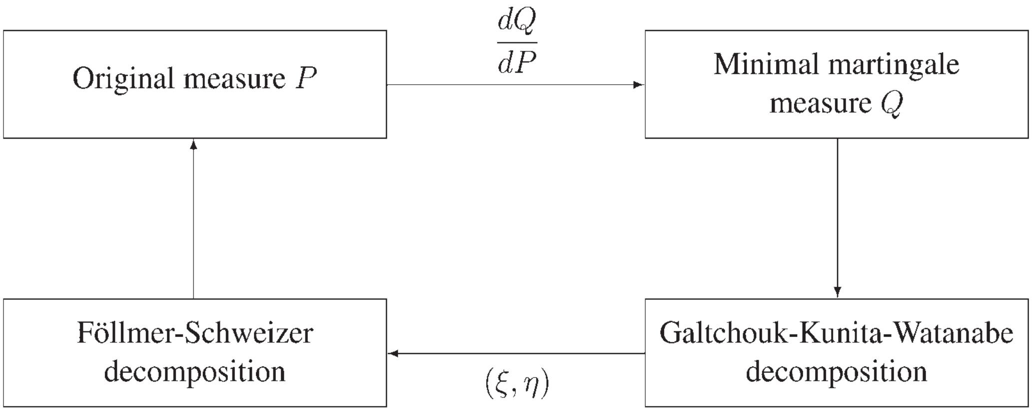

Figure 2: Scheme to determine the locally risk-minimizing hedging strategy in the case of a continuous underlying.

determine the hedging strategy by backward induction. Schäl (1994) proved equivalence in discrete time between the local risk-minimization and the risk-minimization used here, under certain conditions. In continuous time, the former definition is rarely used. The most important application of the theory of Föllmer and Sondermann (1986) is given in Møller (1998). He was the first to apply the risk-minimizing hedging strategy to unit-linked life insurance contracts. He considered a risky asset that followed a Brownian motion and a single premium contract with only a pure endowment and with a term insurance for which all the payments were deferred to maturity of the contract. To be able to hedge intermediate payments, Møller (2001) extended the theory of risk-minimizing hedging strategies. He determined the hedging strategy in the concrete case of a geometric Brownian motion, but the proof is more general in that it holds for every risky asset which is a local martingale.

In fact in Møller (1998), the Brownian motion is only a semimartingale under the original measure, but it suffices to determine the risk-minimizing hedging strategy under the minimal martingale measure when the price process of the risky asset is continuous. Indeed, for the continuous case the locally risk-minimizing hedging strategy under the original measure is equivalent to the risk-minimizing hedging strategy under the minimal martingale measure.

Riesner (2006) wanted to extend the theory of Møller (1998) to the case of a geometric Lévy process. Unfortunately, the equivalence does no longer hold because of the discontinuity of the Lévy process. In Vandaele and Vanmaele (2008a) a correction is given based on Colwell and Elliott (1993), who were the first to show how one can determine the Föllmer-Schweizer decomposition and the related locally risk-minimizing hedging strategy for a contingent claim when the underlying asset follows a Markov diffusion process with jumps.

Furthermore, the locally risk-minimizing hedging strategy for payment streams is defined in Riesner (2007).

Colwell et al. (2007) applied the locally risk-minimizing hedging strategy for index tracking. In Di Masi et al. (1994) various applications of locally risk-minimizing hedging strategies are given for stochastic volatility models. Using the same model for the underlying risky asset, Heath et al. (2001a and 2001b) compared the mean-variance hedging to the locally risk-minimizing strategy theoretically and numerically.

The risk-minimizing hedging strategy under restricted information is investigated by Di Masi et al. (1995) and Schweizer (1993b). The application to stochastic volatility models is studied by Fisher et al. (1999). The extention of this application to the case of semimartingales is investigated by Frey and Runggaldier (1999).
Biagini and Pratelli (1999) proved the invariance under a change of measure of the locally risk-minimizing hedging strategy. Biagini and Cretarola (2007, 2006a, 2006b) determine the locally risk-minimizing hedging strategy for defaultable claims when the risky asset follows a Brownian motion. Becchere and Mulinacci (1999) searched for the locally risk-minimizing hedging strategy to hedge American options in Merton's model.
In Mercurio and Vorst (1997) and in Lamberton et al. (1998), the locally risk-minimizing hedging strategy is defined in discrete time when transaction costs are taken into account.
Finally, we also refer to Møller (2003) where various methods for the hedging and valuation of insurance claims with an inherent financial risk are reviewed. In particular, the emphasis lies on the application to unit-linked life insurance contracts.

4. EXTENSIONS OF THE LOCALLY RISK-MINIMIZING HEDGING STRATEGY

Until now, the locally risk-minimizing hedging strategy has only been defined in the one-dimensional case. If we want to extend it to the multidimensional case, we have to change the definition of locally risk-minimizing hedging strategy. First of all, we define (M) as the sum of all components $\langle M^i, M^i \rangle$:

$$(M) = \sum_{i \in 1,\ldots,d} \langle M^i, M^i \rangle \tag{5}$$

and we also look at the product measure $P_M = P \times (M)$ on the product space $\Omega \times [0,T]$. The extension of the definition of trading strategy and small perturbation to the d-dimensional case is straightforward.

Definition 4.1 (Locally risk-minimizing hedging strategy) *For a trading strategy φ, a small perturbation Δ and a partition τ of $[0,T]$, the risk-quotient $r^\tau[\varphi, \Delta]$ is defined as follows*

$$r^\tau[\varphi, \Delta](\omega, t) := \sum_{t_i, t_{i+1} \in \tau} \frac{R_{t_i}(\varphi + \Delta|_{(t_i, t_{i+1}]}) - R_{t_i}(\varphi)}{E[(M)_{t_{i+1}} - (M)_{t_i} | \mathcal{F}_{t_i}]}(\omega) \mathbb{1}_{(t_i, t_{i+1}]}(t). \tag{6}$$

The strategy φ is then locally risk-minimizing if $\liminf_{n \to \infty} r^{\tau_n}[\varphi, \Delta] \geq 0$ P_M-a.e. for every small perturbation Δ and every increasing 0-convergent sequence (τ_n) of partitions of $[0,T]$.

For the proofs we refer to Vandaele and Vanmaele (2007).
In independent work, Schweizer also extended the one-dimensional case to the multidimensional one. The basic idea is the same but he weakened the restrictions on the risky asset by a slightly more general definition for (M) (5), see Schweizer (2008). At the same time, he extended the strategy for claims to payment streams.
Furthermore, the minimal martingale measure is already defined in the multidimensional case (e.g. Schweizer (1995) and Jeanblanc et al. (2007)) and is the straightforward extension of the

measure in the one-dimensional case. This allows us to easily find the Föllmer-Schweizer decomposition for a continuous risky asset. As in the one-dimensional case, we determine the Galtchouk-Kunita-Watanabe decomposition of the claim under the minimal martingale measure. Since the martingale property and the orthogonality are preserved by the change of measure to the minimal martingale measure, the Föllmer-Schweizer decomposition can be found from the components of the Galtchouk-Kunita-Watanabe decomposition.

The equivalence used before in the continuous case, does no longer hold in the discontinuous case. We show this explicitly in case the underlying risky asset follows a geometric Lévy semimartingale process under the original measure:

$$dX_t = X_{t-}[\alpha_t dt + c\sigma_t dW_t + \int_{\mathbb{R}} \sigma_t x[N(dt, dx) - \nu(dx)dt]].$$

The equivalence between the Galtchouk-Kunita-Watanabe decomposition under the minimal martingale measure and the Föllmer-Schweizer decomposition under the original measure holds if every square-integrable Q-martingale orthogonal to the risky asset under the minimal martingale measure Q is also a martingale under P orthogonal to the martingale part of the discounted risky asset.

We prove that even the martingale property is not preserved and thus it is impossible to talk about the orthogonality between martingales.

We denote by D the Girsanov density describing the change of measure from the minimal martingale measure Q to the original measure P. Assume L is a Q-martingale orthogonal to X under Q, then L is a P-local martingale if and only if L is orthogonal to D under Q. In Vandaele and Vanmaele (2008b) we proved that D is given by

$$D_t = 1 + \int_0^t D_{s-} X_{s-} c\sigma_s \beta_s dW_s^Q + \int_0^t \int_{\mathbb{R}} D_{s-} \frac{X_{s-}\beta_s \sigma_s x}{1 - X_{s-}\beta_s \sigma_s x} [N(ds, dx) - \nu_s^Q(dx)ds]$$

with

$$\beta_t = \frac{\alpha_t}{X_{t-}\sigma_t^2 (c^2 + \int_{\mathbb{R}} x^2 \nu(dx))}.$$

If we now use the condition that L should be orthogonal to X under Q, then L can only be orthogonal to D if L equals the zero process or if we have a trivial change of measure.

We can conclude that the martingale property is not preserved in the case that the underlying risky asset is discontinuous, but this does not mean that it is impossible to determine the Föllmer-Schweizer decomposition. For the case of a geometric Lévy process, we showed already in Vandaele and Vanmaele (2008a) how to find the Föllmer-Schweizer decomposition.

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De handelingen van deze contactfora vormen een aparte publicatiereeks van de Academie.

Contactforum "Actuarial and Financial Mathematics Conference" (7-8 februari 2008, Prof. M. Vanmaele)

Het tweedaags contactforum "Actuarial and Financial Mathematics Conference. Interplay between Finance and Insurance" is een vervolg op de contactfora "Actuarial and Financial Mathematics Day". Deze editie verwelkomde 8 internationaal gerenommeerde gastsprekers en bood aan 9 doctoraats- en post-doc-studenten de mogelijkheid om hun werk voor te stellen dat aansloot bij de onderwerpen van de gastsprekers en met focus op de interactie tussen financieel en actuarieel wiskundige technieken. Discussanten waren aangeduid om de voorgestelde artikels te bespreken en de discussie op gang te trekken. Niet alleen academici uit binnen- en buitenland maar ook heel wat collega's uit de bank- en verzekeringswereld vonden de weg naar dit evenement. Het is de gelegenheid bij uitstek om op de hoogte te blijven van het recente onderzoek op het vlak van financiële en actuariële wiskunde. In deze publicatie vindt u een neerslag van een aantal voorgestelde onderwerpen : de bijdragen betreffen "option pricing", "hedging strategies", "credit default", "optimal regulation", and "operational risk".