KONINKLIJKE VLAAMSE ACADEMIE VAN BELGIE
VOOR WETENSCHAPPEN EN KUNSTEN

ACTUARIAL AND FINANCIAL MATHEMATICS CONFERENCE

Interplay between Finance and Insurance

February 5-6, 2009

Michèle Vanmaele, Griselda Deelstra, Ann De Schepper, Jan Dhaene & Paul Van Goethem (Eds.)

CONTACTFORUM

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Handelingen van het contactforum "Actuarial and Financial Mathematics Conference. Interplay between Finance and Insurance" (5-6 februari 2009, hoofdaanvrager: Prof. M. Vanmaele, UGent) gesteund door de Koninklijke Vlaamse Academie van België voor Wetenschappen en Kunsten.

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Actuarial and Financial Mathematics Conference
Interplay between finance and insurance

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PREFACE

On February 5 and 6, the contactforum “Actuarial and Financial Mathematics Conference” (AFMathConf2009) took place in the buildings of the Royal Flemish Academy of Belgium for Science and Arts in Brussels. The main goal of this conference is to strengthen the ties between researchers in actuarial and financial mathematics from Belgian universities and from abroad on the one side, and professionals of the banking and insurance business on the other side. The conference attracted more than 100 participants from 14 different countries, illustrating the large interest from academia as well as from practitioners.

For the 2009 edition, we slightly changed the formula: next to guest speakers and contributions, the conference also included two short courses and a poster session. During the first day, we welcomed two invited speakers: *Arne Sandström (Swedish Insurance Foundation, Sweden) and Wim Schoutens (Katholieke Universiteit Leuven, Belgium)*, who gave first-class short courses on *Solvency II* and on *Lévy processes*. On the second day, all attendants had the opportunity to listen to four guest speakers: *Ernst Eberlein (University of Freiburg, Germany), Damiano Brigo (Imperial College London & Fitch Solutions, UK and member of the Advisory Board of AMaMeF), Michel Denuit (Université catholique de Louvain, Belgium) and Anna Rita Bacinello (University of Trieste, Italy)*, and to four more contributions from *Alexander van Haastrecht (University of Amsterdam & Delta Lloyd, the Netherlands), Roger Laeven (Tilburg University, the Netherlands), Francesco Menoncin (Università di Brescia, Italy) and Beatrice Acciaio (Vienna University of Technology, Austria)*. We thank them all for their enthusiasm and their interesting presentations which made the conference a great success.

The present proceedings give a nice overview of the activities at the conference. They contain the lecture notes for one of the short courses, two papers corresponding to contributed talks, and ten abstracts of posters presented during the poster sessions on both conference days.

We are much indebted to the members of the scientific committee, Freddy Delbaen (ETH Zurich, Switzerland), Rob Kaas (University of Amsterdam, the Netherlands), Ragnar Norberg (London School of Economics, UK), Bernt Øksendal (University of Oslo, Norway), Antoon Pelsser (University of Amsterdam, the Netherlands), Noel Veraverbeke (Universiteit Hasselt, Belgium) and Griselda Deelstra (Université Libre de Bruxelles & Vrije Universiteit Brussel, Belgium), for the excellent scientific support. We also thank Wouter Dewolf (Ghent University, Belgium), for the administrative work.

We cannot forget our sponsors, who made it possible to organise this event in a very agreeable and inspiring environment. We are very grateful to the Royal Flemish Academy of Belgium for Science and Arts, the Research Foundation ─ Flanders (FWO), the Scientific Research Network (WOG) “Fundamental Methods and Techniques in Mathematics”, le Fonds de la Recherche Scientific (FNRS), the KULeuven Fortis Chair in Financial and Actuarial Risk Management, the IAP programme ─ IAP P6/07 (Belgian Scientific Policy), and the ESF program “Advanced Mathematical Methods in Finance” (AMaMeF).

The success of the meeting encourages us to go on with the organisation of this contactforum. We are sure that continuing this event will provide more opportunities to facilitate the exchange of ideas and results in our fascinating research field.

The editors:
Griselda Deelstra
Ann De Schepper
Jan Dhaene
Paul Van Goethem
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The other members of the organising committee:
Pierre Devolder
Steven Vanduffel
Martine Van Wouwe
David Vyncke

SHORT COURSE

THE WORLD OF VG

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Abstract

This note is a practical guide to the use of jump processes in financial modelling. Jumps and extreme events are crucial stylized features and are essential in the modelling of volatile markets. The recent turmoil in the markets have illustrated once more the need for more refined models. We illustrate how the classical models (driven by Brownian motions, cfr. Black-Scholes settings) can be significantly improved by considering the more flexible class of Lévy processes. By doing this extreme event and jumps are introduced in the models and a more reliable pricing and a better assessment of the risk presents can be made. Besides the setting up of the theoretical framework, many attention is paid to the practical aspects. We deal with the basic vanilla pricing, the calibration of the model to given implied volatility surfaces and exotic option pricing by Monte-Carlo methods.

1. THE BLACK-SCHOLES MODEL

This section overviews the most basic and well-known continuous-time, continuous-variable stochastic model for stock prices. An understanding of this is the first step to the understanding of the pricing of options in a more advanced setting.

1.1. The Normal Distribution

The Normal distribution, Normal(μ, σ^2), is one of the most important distributions in many areas. It lives on the real line, has mean $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$. Its characteristic function is given by

$$\phi_{Normal}(u; \mu, \sigma^2) = \exp(\mathrm{i}u\mu) \exp\left(-\frac{\sigma^2 u^2}{2}\right)$$

and the density function is given as

$$f_{Normal}(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

In Figure 1, one sees the typical bell-shaped curve of the density of a standard normal density.

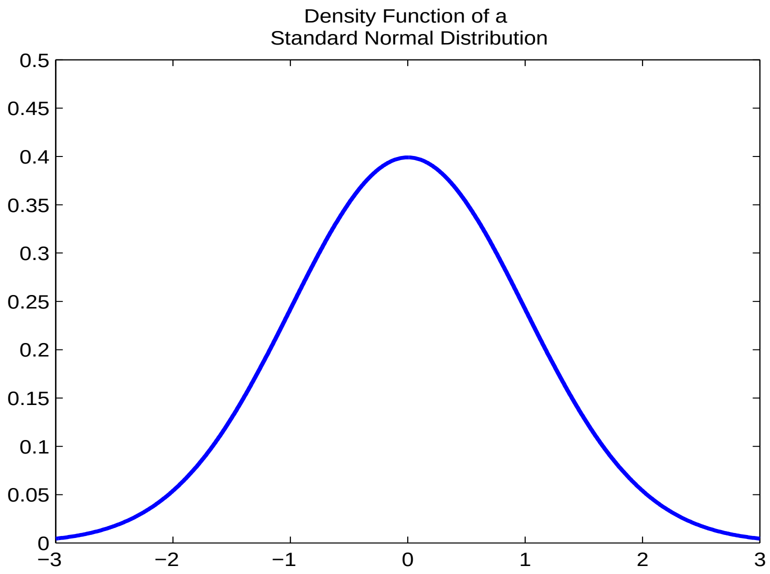

Figure 1: Density of a Standard Normal Distribution

The Normal(μ,σ^2) distribution is symmetric around its mean, and has always a kurtosis equal to 3:

	Normal(μ,σ^2)
mean	μ
variance	σ^2
skewness	0
kurtosis	3

We will denote by

$$\mathrm{N}(x) = \int_{-\infty}^{x} f_{Normal}(u;0,1)\mathrm{d}u \tag{1}$$

the cumulative probability distribution function for a variable that is standard normally distributed (Normal$(0,1)$). This special function is available in most mathematical software packages.

1.2. Brownian Motion

The big brother of the Normal distribution is the Brownian motion. Brownian motion is the dynamic counterpart – where we work with evolution in time – of its static counterpart, the Normal distribution. Both arise from the central limit theorem. Intuitively, it tells us that the suitable normalized sum of many small independent random variables is approximately normally distributed.

These results explain the ubiquity of the Normal distribution in a static context. If one works in a dynamic setting, i.e. with stochastic processes, Brownian motion appears in the same manner.

1.2.1. The History of Brownian Motion

The history of Brownian motion dates back to 1828, when the Scottish botanist Robert Brown observed pollen particles in suspension under a microscope and observed that they were in constant irregular motion. By doing the same with particles of dust, he was able to rule out that the motion was due to pollen being ”alive”.

In 1900 L. Bachelier considered Brownian motion as a possible model for stock market prices. Bachelier’s model was his thesis. At that time the topic was not thought worthy of study.

In 1905 Albert Einstein considered Brownian motion as a model of particles in suspension. Einstein observed that, if the kinetic theory of fluids was right, then the molecules of water would move at random and so a small particle would receive a random number of impacts of random strength and from random directions in any short period of time. Such a bombardment would cause a sufficiently small particle to move in exactly the way described by Brown. Einstein also used it to estimate Avogadro’s number.

In 1923 Norbert Wiener defined and constructed Brownian motion rigorously for the first time. The resulting stochastic process is often called the Wiener process in his honor.

It was with the work of [104] that Brownian motion reappeared as a modeling tool in finance.

1.2.2. Definition

A stochastic process $X = \{X_t, t \geq 0\}$ is a *standard Brownian motion* on some probability space $(\Omega, \mathcal{F}, P)$, if

1. $X_0 = 0$ a.s.

2. X has independent increments.

3. X has stationary increments.

4. $X_{t+s} - X_t$ is normally distributed with mean 0 and variance $s > 0$: $X_{t+s} - X_t \sim$ Normal$(0, s)$.

We shall henceforth denote standard Brownian motion by $W = \{W_t, t \geq 0\}$ (W for Wiener). Note that the second item in the definition implies that Brownian motion is a Markov process. Moreover Brownian motion is the basic example of a Lévy process (see [113]).

In the above, we have defined Brownian motion without reference to a filtration. Without other notice, we will always work with the natural filtration $\mathbb{F} = \mathbb{F}^W = \{\mathcal{F}_t, 0 \leq t \leq T\}$ of W. We have that Brownian motion is adapted with respect to this filtration and that increments $W_{t+s} - W_t$ are independent of $\mathcal{F}_t$.

1.2.3. Random-Walk Approximation of Brownian Motion

No construction of Brownian motion is easy. We take the existence of Brownian motion for granted. To gain some intuition on its behaviour, it is good to compare Brownian motion with a simple symmetric random walk on the integers. More precisely, let $X = \{X_i, i = 1, 2, \dots\}$ be a series of independent and identically distributed random variables with $P(X_i = 1) = P(X_i = -1) = 1/2$. Define the simple symmetric random walk $Z = \{Z_n, n = 0, 1, 2, \dots\}$ as $Z_0 = 0$ and $Z_n = \sum_{i=1}^{n} X_i$, $n = 1, 2, \dots$. Rescale this random walk as $Y_k(t) = Z_{\lfloor kt \rfloor}/\sqrt{k}$, where $\lfloor x \rfloor$ is the integer part of x. Then from the Central Limit Theorem, $Y_k(t) \to W_t$ as $k \to \infty$, with convergence in distribution (or weak convergence).

In Figure 2, one sees a realization of the standard Brownian motion. In Figure 3, one sees the

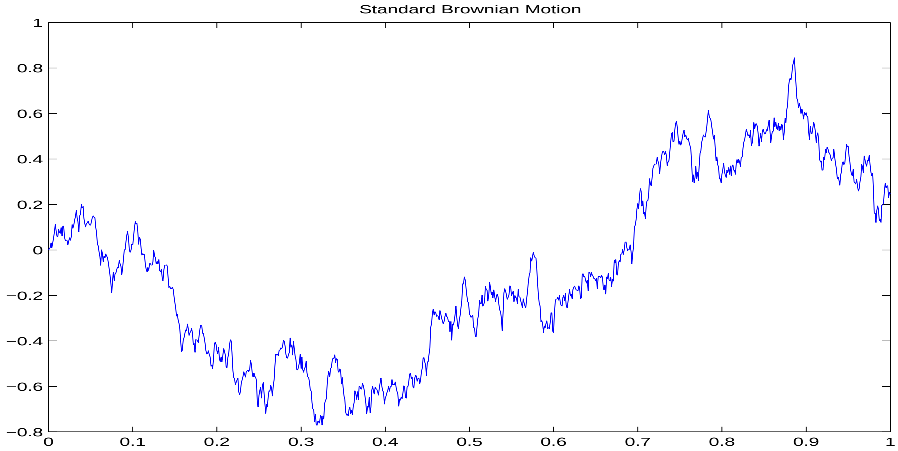

Figure 2: A sample path of a standard Brownian motion

random-walk approximation of the standard Brownian motion. The process $Y_k = \{Y_k(t), t \geq 0\}$ is shown for $k = 1$ (i.e. the symmetric random walk), $k = 3$, $k = 10$ and $k = 50$. Clearly, one sees the $Y_k(t) \to W_t$.

1.2.4. Properties

Next, we look at some of the classical properties of Brownian motion.

Martingale Property Brownian motion is one of the most simple examples of a martingale. We have for all $0 \leq s \leq t$,

$$E[W_t|\mathcal{F}_s] = E[W_t|W_s] = W_s.$$

We also mention that one has:

$$E[W_t W_s] = \min\{t, s\}.$$

Path Properties One can proof that Brownian motion has continuous paths, i.e. W_t is a continuous function of t. However the paths of Brownian motion are very erratic. They are for example

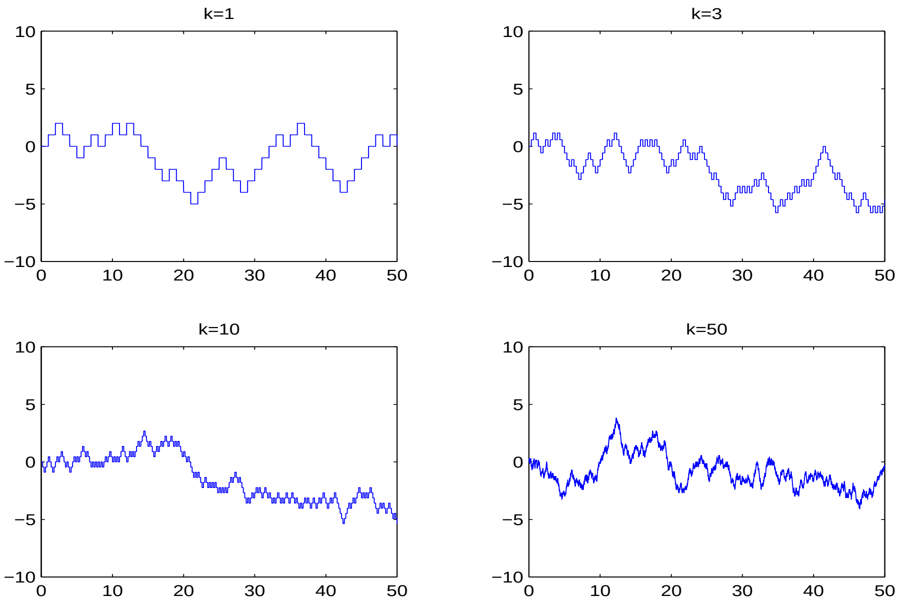

Figure 3: Random walk approximation for standard Brownian motion

nowhere differentiable. Moreover, one can prove also that the paths of Brownian motion are of infinite variation, i.e. their variation is infinite on every interval.

Another property is that for a Brownian motion $W = \{W_t, t \geq 0\}$, we have that

$$P(\sup_{t\geq 0} W_t = +\infty \text{ and } \inf_{t\geq 0} W_t = -\infty) = 1.$$

This result tells us that the Brownian path will keep oscillating between positive and negative values.

Scaling Property There is a well-known set of transformations of Brownian motion which produce another Brownian motion. One of this is the scaling property which says that if $W = \{W_t, t \geq 0\}$ is a Brownian motion, then also for every $c \neq 0$,

$$\tilde{W} = \{\tilde{W}_t = cW_{t/c^2}, t \geq 0\} \quad (2)$$

is a Brownian motion.

1.3. Geometric Brownian Motion

Now that we have Brownian motion W, we can introduce an important stochastic process for us, a relative of Brownian motion – *geometric Brownian motion*.

In the Black-Scholes model, one models the time evolution of a stock price $S = \{S_t, t \geq 0\}$ as follows. Consider how S will change in some small time interval from the present time t to a time $t + \Delta t$ in the near future. Writing ΔS_t for the change $S_{t+\Delta t} - S_t$, the return in this interval is

$\Delta S_t/S_t$. It is economically reasonable to expect this return to decompose into two components, a *systematic* part and a *random* part.

Let us first look at the systematic part. One assumes that the stock's expected return over a period is proportional with the length of the period considered. This means that in a short interval of time $[S_t, S_{t+\Delta t}]$ of length Δt, the expected increase in S is given by $\mu S_t \Delta t$, where μ is some parameter representing the mean rate of the return of the stock. In other words, the deterministic part of the stock return is modeled by $\mu \Delta t$.

A stock price fluctuates stochastically, and a reasonable assumption is that the variance of the return over the interval of time $[S_t, S_{t+\Delta t}]$ is proportional to the length of the interval. So, the random part of the return is modeled by $\sigma \Delta W_t$, where ΔW_t represents the (normally distributed) noise term (with variance Δt) driving the stock price dynamics, and $\sigma > 0$ is the parameter which describes how much effect the noise has – how much the stock price fluctuates. In total the variance of the return equals $\sigma^2 \Delta t$. Thus σ governs how volatile the price is, and is called the *volatility* of the stock. Putting this together, we have

$$\Delta S_t = S_t(\mu \Delta t + \sigma \Delta W_t), \quad S_0 > 0.$$

In the limit, as $\Delta t \to 0$, we have the stochastic differential equation (SDE)

$$\mathrm{d}S_t = S_t(\mu \mathrm{d}t + \sigma \mathrm{d}W_t), \quad S_0 > 0. \tag{3}$$

The stochastic differential equation above has the unique solution

$$S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t\right).$$

This (exponential) functional of Brownian motion is called geometric Brownian motion. Note that

$$\log S_t - \log S_0 = \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t$$

has a Normal($t(\mu - \sigma^2/2), \sigma^2 t$) distribution. Thus S_t itself has a *lognormal* distribution. This geometric Brownian motion model, and the log-normal distribution which it entails, are the basis for the Black-Scholes model for stock-price dynamics in continuous time.

In Figure 4, one sees the realization of the geometric Brownian motion based on the sample path of the standard Brownian motion of Figure 2.

1.4. The Black-Scholes Option Pricing Model

In the early 1970s, Fischer Black, Myron Scholes, and Robert Merton made a major breakthrough in the pricing of stock options by developing what has become known as the Black-Scholes model. The model has had huge influence on the way that traders price and hedge options. In 1997, the importance of the model was recognized when Myron Scholes and Robert Merton were awarded the Nobel prize for economics. Sadly, Fischer Black died in 1995, otherwise he also would undoubtedly have been one of the recipients of this prize.

We show how the Black-Scholes model for valuing European call and put options on a stock works.

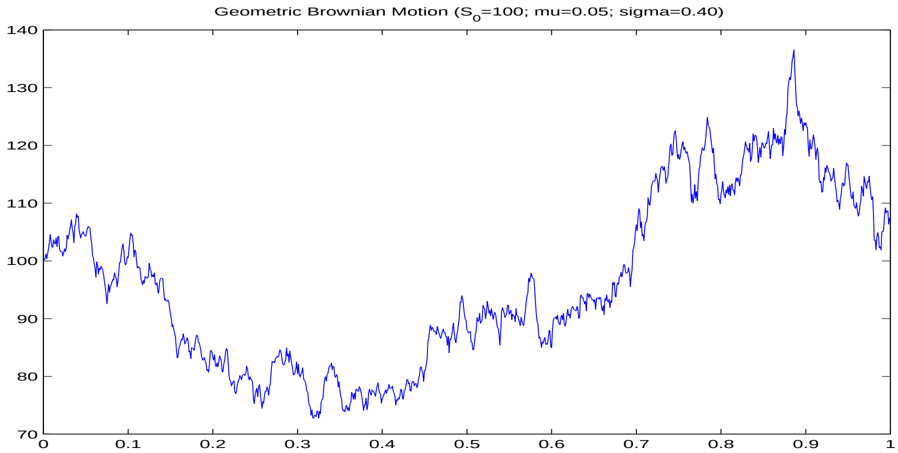

Figure 4: Sample path of a geometric Brownian motion ($S_0 = 100, \mu = 0.05, \sigma = 0.40$)

1.4.1. The Black-Scholes Market Model

Investors are allowed to trade continuously up to some fixed finite planning horizon T. The uncertainty is modeled by a filtered probability space $(\Omega, \mathcal{F}, P)$. We assume a frictionless market with 2 assets.

The first asset is one without risk (the bank account). Its price process is given by $B = \{B_t = \exp(rt), 0 \leq t \leq T\}$. The second asset is a risky asset, usually referred to as stock, and which pays a continuous dividend yield $q \geq 0$. The price process of this stock, $S = \{S_t, 0 \leq t \leq T\}$, is modeled by geometric Brownian motion:

$$B_t = \exp(rt), \qquad S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t\right),$$

where $W = \{W_t, t \geq 0\}$ is standard Brownian motion.

Note that, under P, W_t has a Normal$(0, t)$ and that $S = \{S_t, t \geq 0\}$ satisfies the SDE (3). The parameter μ is reflecting the drift and σ models the volatility; μ and σ are assumed to be constant over time.

We assume as underlying filtration, the natural filtration $\mathbb{F} = (\mathcal{F}_t)$ generated by W. Consequently, the stock price process $S = \{S_t, 0 \leq t \leq T\}$ follows a strictly positive adapted process. We call this market model the *Black-Scholes model*. It is a well-established result that the Black-Scholes model is a complete model, that is, every contingent claim can be replicated by a dynamic self-financing trading strategy.

1.4.2. The Risk-Neutral Setting

Since the Black-Scholes market model is complete there exists only one equivalent martingale measure Q. It is not hard to see that under Q, the stock price is following a Geometric Brownian motion again (Girsanov theorem). This risk-neutral stock price process has the same volatility parameter σ, but the drift parameter μ is changed to the continuously compounded risk-free rate r

minus the dividend yield q:

$$S_t = S_0 \exp\left(\left(r - q - \frac{\sigma^2}{2}\right) t + \sigma W_t\right).$$

Equivalent, we can say that under Q our stock price process $S = \{S_t, 0 \leq t \leq T\}$ is satisfying the SDE:

$$\mathrm{d}S_t = S_t((r-q)\mathrm{d}t + \sigma \mathrm{d}W_t), \quad S_0 > 0.$$

This SDE tells us that in a risk-neutral world the total return from the stock must be r; the dividends provide a return of q, the expected growth rate in the stock price, therefore, must be $r - q$.

Next, we will calculate European call option prices under this model.

1.4.3. The Pricing of Options under the Black-Scholes Model

General Pricing Formula By the risk-neutral valuation principle the price V_t at time t, of a contingent claim with payoff function $G(\{S_u, 0 \leq u \leq T\})$ is given by

$$V_t = \exp(-(T-t)r) E_Q[G(\{S_u, 0 \leq u \leq T\}) | \mathcal{F}_t], \quad t \in [0, T]. \tag{4}$$

Furthermore, if the payoff function is only depending on the time T value of the stock, i.e. $G(\{S_u, 0 \leq u \leq T\}) = G(S_T)$, then the above formula can be rewritten as (we set for simplicity $t = 0$):

$$\begin{aligned} V_0 &= \exp(-Tr) E_Q[G(S_T)] \\ &= \exp(-Tr) E_Q[G(S_0 \exp((r - q - \sigma^2/2)T + \sigma W_T))] \\ &= \exp(-Tr) \int_{-\infty}^{+\infty} G(S_0 \exp((r - q - \sigma^2/2)T + \sigma x)) f_{Normal}(x; 0, T) \mathrm{d}x. \end{aligned}$$

Black-Scholes PDE If moreover $G(S_T)$ is a sufficiently integrable function, then the price is also given by $V_t = F(t, S_t)$, where F solves the *Black-Scholes partial differential equation*

$$\begin{aligned} \frac{\partial}{\partial t} F(t,s) + (r-q)s \frac{\partial}{\partial s} F(t,s) + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2}{\partial s^2} F(t,s) - rF(t,s) &= 0, \\ F(T,s) &= G(s) \end{aligned} \tag{5}$$

This follows from the Feynman-Kac representation for Brownian motion (see e.g. [24]).

Explicit Formula for European Call and Put Options Solving the Black-Scholes partial differential equation (5) is not always that easy. However, in some cases it is possible to evaluate explicitly the above expected value in the risk-neutral pricing formula (4).

Take for example an European call on the stock (with price process S) with strike K and maturity T (so $G(S_T) = (S_T - K)^+$). The Black-Scholes formulas for the price $C(K, T)$ at time zero of this European call option on the stock (with dividend yield q) is given by

$$C(K,T) = C = \exp(-qT) S_0 \mathrm{N}(d_1) - K \exp(-rT) \mathrm{N}(d_2),$$

where

$$d_1 = \frac{\log(S_0/K) + (r - q + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}, \quad (6)$$

$$d_2 = \frac{\log(S_0/K) + (r - q - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}, \quad (7)$$

and $\mathrm{N}(x)$ is the cumulative probability distribution function for a variable that is standard normally distributed (Normal$(0, 1)$).

From this, one can also easily (via the put-call parity) obtain the price $P(K, T)$ of the European put option on the same stock with same strike K and same maturity T :

$$P(K, T) = -\exp(-qT)S_0\mathrm{N}(-d_1) + K\exp(-rT)\mathrm{N}(-d_2).$$

For the call, the probability (under Q) of finishing in the money corresponds with $\mathrm{N}(d_2)$. Similarly, the delta (i.e. the change in the value of the option compared with the change in the value of the underlying asset) of the option corresponds with $\mathrm{N}(d_1)$.

2. SHORTFALLS OF BLACK-SCHOLES

Over the last decades the Black-Scholes model

$$S_t = S_0\exp((\mu - \sigma^2/2)t + \sigma W_t), \qquad t \geq 0,$$

where $\{W_t, t \geq 0\}$ is standard Brownian Motion and σ is the usual volatility, turned out to be very popular. One should bear in mind however, that this elegant theory hinges on several crucial assumptions. We assume that there are no market frictions, like taxes and transaction costs or constraints on the stock holding, etc. Moreover, empirical evidence suggests that the classical Black-Scholes model does not describe the statistical properties of financial time series very well.

Summarizing we could say that under the Black-Scholes framework the following problems have serious impact on the modeling of financial assets and the corresponding pricing and hedging of financial derivatives:

- log-returns under the Black-Scholes model are Normally distributed. However it is observed from empirical data that log-returns typically do not behave according to a Normal distribution. They show most of the time negative skewness and excess kurtosis.

- related to the above observation on the log-returns, the Black-Scholes model can not model realistically extreme events.

- paths of the stock process under the Black-Scholes model are continuous and show no jumps. However in reality one could say that everything is driven by jumps. Moreover, it are especially the more pronounced jumps that have typically the most impact for the derivative pricing under question.

- the volatility parameter (the only model parameter of relevance for the pricing of derivatives) is assumed to be constant. However, it has been observed that the volatilities or the parameters of uncertainty estimated (or more generally the environment) change stochastically over time and are clustered.

Next, we will focus on each of the above problems a bit more in detail.

2.1. Normal Returns

In Table 1 we summarize i.a. the empirical mean, standard deviation, skewness and kurtosis for a set of popular indices. The first data set (SP500 (1970-2001)) contains all daily log-returns of the SP500 index over the period 1970-2001. The second data set (*SP500 (1970-2001)) contains the same data except the exceptional log-return (-0.2290) of the crash on the 19th of October 1987. All other data sets are over the period 1997-1999.

2.1.1. Skewness, Kurtosis and Fait-Tails

We note that the skewness measures the degree to which a distribution is asymmetric and is defined to be the third moment about the mean, divided by the third power of the standard deviation:

$$\frac{E[(X-\mu_X)^3]}{\text{Var}[X]^{3/2}}$$

For a symmetric distribution (like the Normal(μ, σ^2)), the skewness is zero. If a distribution has a longer tail to the left than to the right, it is said to have negative skewness. If the reverse is true, then the distribution has a positive skewness. If we look at the daily log-returns of the different indices, we observe typically some significant (negative) skewness.

Tail behavior and peakedness are measured by kurtosis, which is defined by

$$\frac{E[(X-\mu_X)^4]}{\text{Var}[X]^2}.$$

For the Normal distribution (mesokurtic), the kurtosis is 3. If the distribution has a flatter top (platykurtic), the kurtosis is less than 3. If the distribution has a high peak (leptokurtic), the kurtosis is greater than 3.

We clearly see that our data always gives rise to a kurtosis clearly bigger than 3, indicating that the tails of the Normal distribution go much faster to zero than the empirical data suggests and that the distribution is much more peaked. So large asset price movements occur more frequently than in a model with Normal distributed increments. This feature is often referred to as *excess kurtosis* or *fat tails*; it is one of the main reasons for considering asset price processes with jumps. The fact that return distributions are more leptokurtic than the Normal one was already noted by [46].

2.1.2. Kernel Density Estimation

Next, we look at the empirical density of daily log-returns.

Index	**Mean**	**St.Dev.**	**Skewness**	**Kurtosis**
SP500 (1970-2001)	0.0003	0.0099	-1.6663	43.36
*SP500 (1970-2001)	0.0003	0.0095	-0.1099	7.17
SP500 (1997-1999)	0.0009	0.0119	-0.4409	6.94
Nasdaq-Composite	0.0015	0.0154	-0.5439	5.78
DAX	0.0012	0.0157	-0.4314	4.65
SMI	0.0009	0.0141	-0.3584	5.35
CAC-40	0.0013	0.0143	-0.2116	4.63

Table 1: Mean, standard deviation, skewness and kurtosis of major indices

In order to estimate the empirical density, we make use of kernel density estimators. The goal of density estimation is to approximate the probability density function $f(x)$ of a random variable X. Assume we have n independent observations $x_1, \dots, x_n$ from the random variable X. The *kernel density estimator* $\hat{f}_h(x)$ for the estimation of the density $f(x)$ at point x is defined as

$$\hat{f}_h(x) = \frac{1}{nh} \sum_{i=1}^{n} K\left(\frac{x_i - x}{h}\right),$$

where $K(x)$ is a so-called kernel function, and h is the bandwidth. We typically work with the so-called Gaussian kernel: $K(x) = \exp(-x^2/2)/\sqrt{2\pi}$. Other possible kernel functions are the so-called Uniform, Triangle, Quadratic and cosine kernel function. In the above formula one also has to select the bandwidth h. We use with our Gaussian kernel, Silverman's rule of thumb value $h = 1.06\sigma n^{-1/5}$.

In Figure 5, one sees the Gaussian kernel density estimator based on the daily log-returns of the SP500 Index over the period 1970 until end 2001. We see a sharp peaked distribution. This tell us that most of the time stock prices do not move that much; there is a considerable amount of mass around zero. Also in Figure 5 we plotted the Normal density with mean $\mu = 0.0003112$ and $\sigma = 0.0099$ corresponding to the empirical mean and standard deviation of the daily log-returns.

2.1.3. Semi-Heavy Tails

Density plots focus on the center of the distribution, however also the tail behavior is important. Therefore, we show in Figure 5 the log densities, i.e. $\log \hat{f}_h(x)$ and the corresponding log of the Normal density. The log-density of a Normal distribution has a quadratic decay, whereas the empirical log-density seems to have a much more linear decay. This feature is typical for financial data and is often referred to as the semi-heaviness of the tails. We say that a distribution or its density function $f(x)$ has semi-heavy tails, if the tails of the density function behave as

$$\begin{aligned} f(x) &\sim C_-|x|^{\rho_-}\exp(-\eta_-|x|) \quad \text{as} \quad x \to -\infty \\ f(x) &\sim C_+|x|^{\rho_+}\exp(-\eta_+|x|) \quad \text{as} \quad x \to +\infty, \end{aligned}$$

for some $\rho_-, \rho_+ \in \mathbb{R}$ and $C_-, C_+, \eta_-, \eta_+ \geq 0$. Equivalently,

$$\begin{aligned} \log f(x) &\sim A_- \log|x| - \eta_-|x| \quad \text{as} \quad x \to -\infty \\ \log f(x) &\sim B_+ \log|x| - \eta_+|x| \quad \text{as} \quad x \to +\infty, \end{aligned}$$

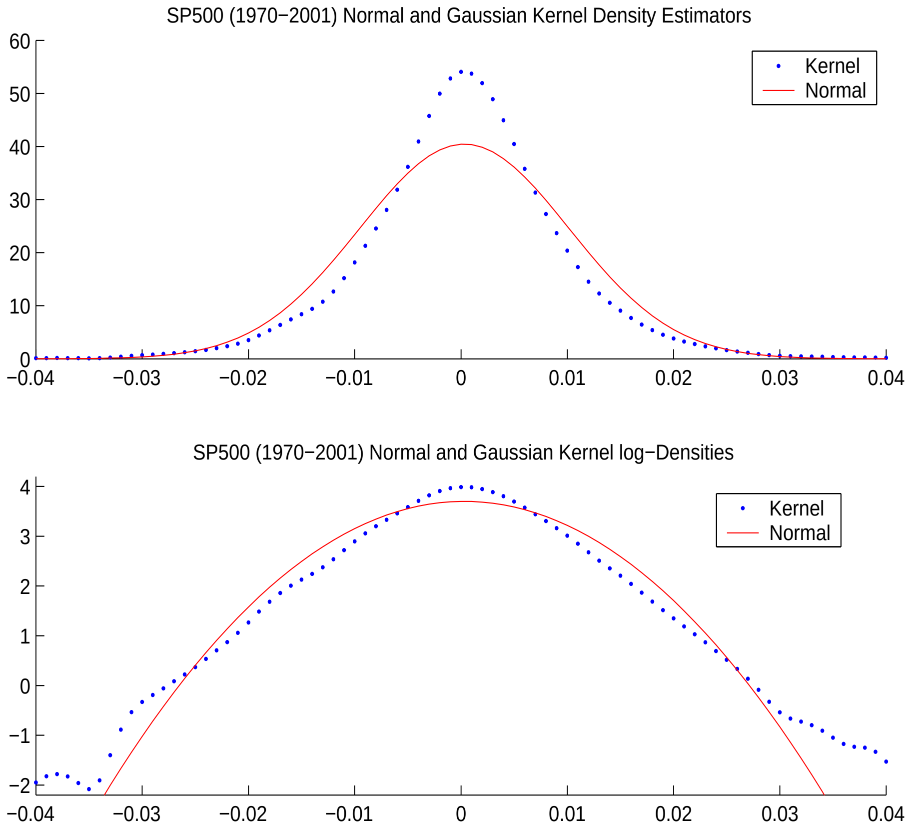

Figure 5: Normal density and Gaussian Kernel estimator of the density of the daily log-returns of the SP500 index

for some $A_-, B_+ \in\in \mathbb{R}$ and $\eta_-, \eta_+ \geq 0$. The log-densities for semi-heavy distributions and apparently also financial returns show a linear behavior of the tails towards infinity.

The Normal distribution with mean μ and variance σ^2 exhibits a quadratic decay near infinity of the logarithm of its probability density function:

$$\log f_{\text{Normal}}(x;\mu,\sigma^2) = -\frac{(x-\mu)^2}{2\sigma^2} - \log\left(\sigma\sqrt{2\pi}\right) \sim -\frac{1}{2\sigma^2}x^2 \tag{8}$$

as $x \to \pm\infty$. In conclusion, we clearly see that the Normal distribution leads to a very bad fit.

2.1.4. Statistical Testing

All the above is confirmed by statistical tests on the Normal hypotheses. A standard and straightforward way of testing goodness of fit of a distribution can be done with the so-called χ^2-test. The χ^2-test counts the number of sample points falling into certain intervals and compares them with the expected number under the null hypothesis.

More precisely, suppose we have n independent observations $x_1, \ldots, x_n$ from the random variable X and we want to test whether these observations follow a law with distribution D, depending on h parameters which we all estimate by some method. First, make a partition $\mathcal{P} = \{A_1, \ldots A_m\}$

of the support (in our case $\mathbb{R}$) of D. The classes A_k can be chosen arbitrarily; we consider classes of equal width.

Let N_k, $k = 1, \ldots, m$ be the number of observations x_i falling into the set A_k; N_k/n is called the empirical frequency distribution. We will compare these numbers with the theoretical frequency distribution π_k, defined by

$$\pi_k = P(X \in A_k), \quad k = 1, \ldots, m,$$

through the Pearson statistic

$$\hat{\chi}^2 = \sum_{k=1}^{m} \frac{(N_k - n\pi_k)^2}{n\pi_k}.$$

If necessary we collapse outer cells, such that the expected value $n\pi_k$ of observations becomes always greater than five.

We say a random variable χ_j^2 follows a χ^2-distribution with j degrees of freedom if it has a Gamma$(j/2, 1/2)$ law (see Chapter 5):

$$E[\exp(\mathrm{i}u\chi_j^2)] = (1 - 2\mathrm{i}u)^{-j/2}.$$

General theory says that the Pearson statistic $\hat{\chi}^2$ follows (asymptotically) a χ^2-distribution with $m - 1 - h$ degrees of freedom.

The P-value of the $\hat{\chi}^2$ statistic is defined as

$$P = P(\chi^2_{m-1-h} > \hat{\chi}^2).$$

In words, P is the probability that values are even more extreme (more in the tail) than our test-statistic. It is clear that very small P-values lead to a rejection of the null hypotheses, because they are themselves extreme. P-values not close to zero indicate that the test statistic is not extreme and lead not to a rejection of the hypothesis. To be precise we reject if the P-value is less than our level of significance, which we take equal to 0.05.

Next, we calculate the P-value for the same set of indices. Table 2 shows the P-values of the test-statistics. Similar tests can be found i.a. in [40].

Index	P_{Normal}-value	**Class boundaries**
SP500 (1970-2001)	0.0000	$-0.0300 + 0.0015\, i, \quad i = 0, \ldots, 40$
SP500 (1997-1999)	0.0421	$-0.0240 + 0.0020\, i, \quad i = 0, \ldots, 24$
DAX	0.0366	$-0.0225 + 0.0015\, i, \quad i = 0, \ldots, 30$
Nasdaq-Comp.	0.0049	$-0.0300 + 0.0020\, i, \quad i = 0, \ldots, 30$
CAC-40	0.0285	$-0.0180 + 0.0012\, i, \quad i = 0, \ldots, 30$
SMI	0.0479	$-0.0180 + 0.0012\, i, \quad i = 0, \ldots, 30$

Table 2: Normal χ^2-test: P-values and class boundaries

We see that the Normal hypothesis is always rejected. Basically we can conclude that the Normal distribution, is not sufficiently flexible to capture all features of the data. We need at least four parameters: a location parameter, a scale (volatility) parameter, an asymmetry (skewness) parameter and a (kurtosis) parameter describing the decay of the tails. We will see that the Lévy models introduced in the next chapter will have this required flexibility.

We have just seen that the well-mannered bell curve of the Gaussian distribution isn't so normal at all. Next, we focus a bit more on the impact of this on the extreme events and the corresponding implications of more fatter tails.

2.2. Jumps and Extreme Events

From the above it should be already clear that the stock market doesn't behave according to Normal laws. Finance likes it hotter, spicier, more extreme. Indeed, extreme price swings are more likely than the Black-Scholes incorporates them. This insight is not new. Mandelbrot already elaborated on it in the sixties, long before the Black-Scholes model was ruling Wall Street (see e.g. [60]).

The fact that the problem with the Normal (Gaussian) distribution lies certainly also in the tails is illustrated by looking at the most severe crashes in a fifth years time period. More precisely, we look at the Dow Jones Industrial Average and Table 3 lists the ten largest relative down moves of the Dow over the last fifty years (1954–2004).

Date	**Closing**	**logreturn**	**Average frequency under Normal law**
19-Oct-87	1738.74	-0.2563	once in 10^{53} years
			US: 100 sexdecillion, UK : 100000 octillion
26-Oct-87	1793.93	-0.0838	once in 72503 years
15-Oct-08	8577.91	-0.0820	once in 41318 years
01-Dec-08	8149.09	-0.0801	once in 21725 years
09-Oct-08	8579.19	-0.0762	once in 6068 years
27-Oct-97	7161.15	-0.0745	once in 3402 years
17-Sep-01	8920.7	-0.0740	once in 2914 years
29-Sep-08	10365.45	-0.0723	once in 1798 years
13-Oct-89	2569.26	-0.0716	once in 1405 years
08-Jan-88	1911.31	-0.0710	once in 1173 years

Table 3: Ten largest down moves of the Dow since 1954

Under the Black-Scholes regime, what is the probability that the Dow will suffer a big loss tomorrow? Everything depends of course on the volatility that you plug in. Figure 6 shows the annualized historical volatility estimated on the basis of, say, a three-year window. Clearly, volatility is not constant and behaves stochastically – another point we will come back to shortly. In the figure, volatility is typically below 25%. Let us calculate for a 25% vol the frequency of a negative log-return of -0.0582 or even worse. Under the assumption of Normality, it happens just once every 35 years. In reality, we have witnessed ten in the last 50 years! If the mathematician Thales (c.624–c.546 BC) – one of the ancient derivatives traders – would have been granted eternal live, he would according to the Normal distribution have seen only one down move of -0.0716 or worse up to now. In the last fifty years we had five! A Homo Sapiens would likely have witnessed only one down move of -0.0838 or worse up to now. In a particularly bad month, October 1987, there were two! What is the probability of a down move of -0.25 or worse: It is of the order once in

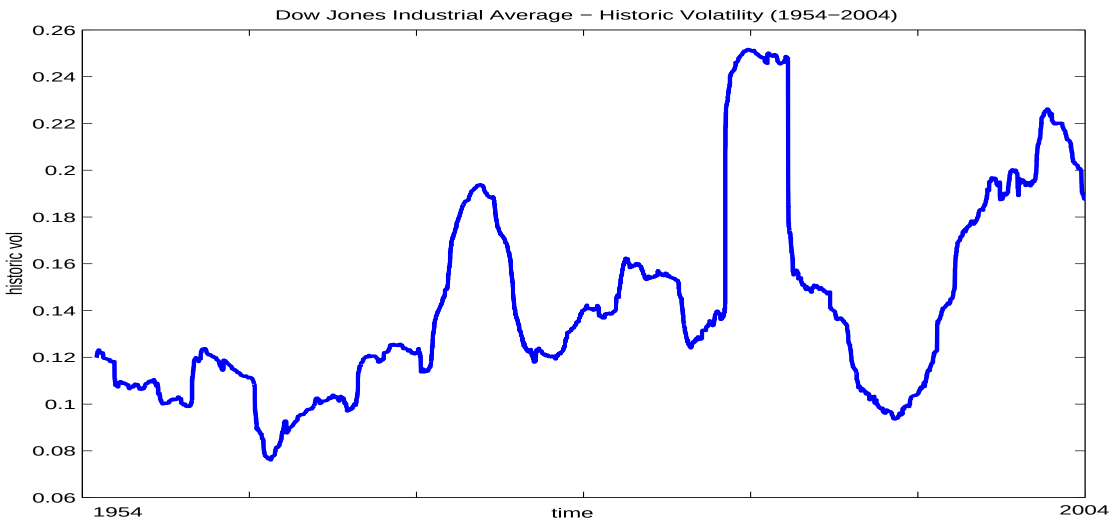

Figure 6: *Dow Jones Industrial Average – Historic Volatility (1954–2004)*

the 10^{53} years (in US language: 100 sexdecillion years, UK language: 100000 octillion years). In contrast, the Big Bang *only* happened around 15×10^9 years ago. The present generation must be really exceptional that God allowed the Dow to crash in October 1987.

2.2.1. Expected Shortfall

Let us focus a bit more on the modeling of extreme values and the tale a tail has to tell. One of the main developer of the theory was the German mathematician, pacifist, and anti-Nazist Emil Julius Gumbel who described the Gumbel distribution in the 1950s (see [55]). Extreme value theory is by now a well-developed area of statistics and finds applications in many areas of research: besides finance, it is/can be used in hydrology, cosmology, insurance, pollution and climatology, geology, etc. A basic reference text is [19].

A risk measure currently gaining in popularity is the *expected shortfall*, defined as the expected excess over a given (high) level, conditionally on this level being exceeded. The sample version of the expected shortfall over a certain level is simply the average of the excesses over that level. The expected shortfall over the 1000 largest negative daily log-returns of the Dow are plotted in Figure 7(a). Note that the expected shortfall is increasing with the level: the higher the level being exceeded, the higher the excess by which it will be exceeded! Once more, this is in sharp contrast with panel (b) of the same figure: for a Normal sample with the same mean and variance, the expected shortfall decreases rapidly (note the different axes). In a light-tailed world, given that you exceed a high level, you hardly exceed it at all. But in a heavy-tailed world, once you know you'll get hit, you may get hit much harder than expected!

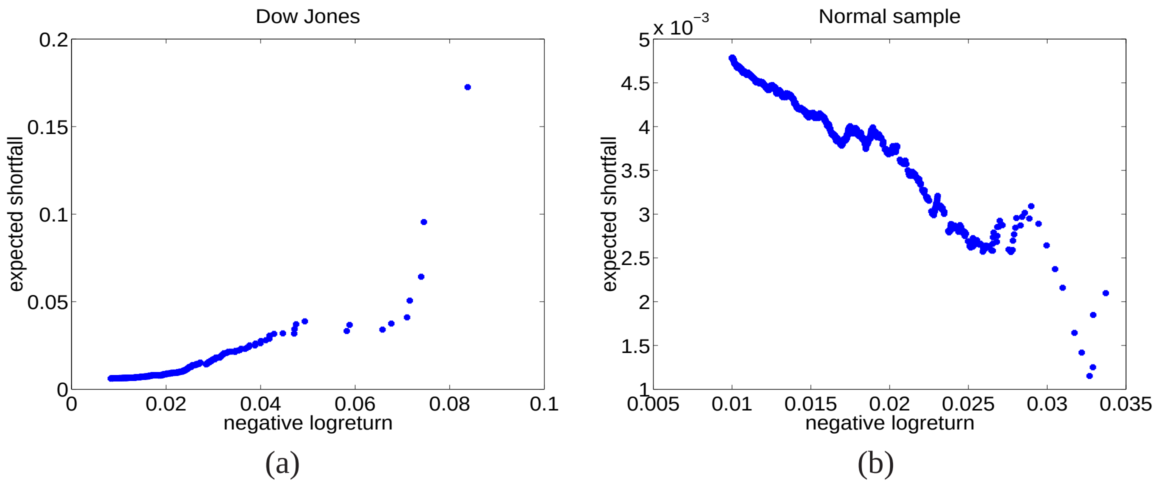

Figure 7: (a) *Expected shortfall over 1000 largest negative daily log-return of the Dow (1954–2004).* (b) *Similarly for a Normal random sample with the same mean and variance.*

2.3. No Jumps

Brownian motion has continuous sample paths, whereas in reality prices are driven by jumps. The Brownian motion needs a substantial amount of time to reach a low barrier, whereas in reality jumps can cause an almost immediate move over the barrier. This has serious impact for example on the pricing of barrier products. Because the probability that on the short-term Brownian motion will hit a barrier far away from its current position is almost zero, prices of down-and-in and up-and-in type of barrier options with short maturities are completely underestimated. Indeed since under Black-Scholes there is almost no possibility that in the short-term the Barrier is hit and thus the options becomes "in" the price of the product will be extremely low. In reality however, we have seen above that even in one day extreme movements are possible and that actually the hitting of the barrier is much more likelier. Processes with jumps incorporate this effect and actually make it possible that even in the next instance the Barrier is trigger. We already here note that this will be especially crucial in Credit Risk modeling, where it are exactly these extreme default events that are of importance. Many of the credit derivatives (like for example the Credit Default Swap) can be seen (under a firm-value model approach) as barrier products with a very low barrier (see for example [114]).

2.4. Volatility

Another important feature which the Black-Scholes model is missing is the fact that volatility or more generally the environment is changing stochastically over time.

2.4.1. Historic Volatility

It has been observed that the volatilities estimated (or more general the parameters of uncertainty) change stochastically over time. This can be seen for example by looking at *historic volatilities*. Historical volatility is a retrospective measure of volatility. It reflects how volatile the asset has been in the recent past. Historical volatility can be calculated for any variable for which historical data is tracked.

For the SP500 index, we estimated for every day from 1971 to 2001 the standard deviation of the daily log-returns over a one year period preceding the day. In Figure 8, we plot, for every day in the mentioned period, the annualized standard deviation, i.e. we multiply the simulated standard deviation with the square root of the number of trading days in one calendar year. Typically, there are around 250 trading days in one year. This annualized standard deviation is called *the historic volatility*. In Figure 6 the historical volatility estimated (using a three-years window) was already given for the Dow Jones Industrial Average. Clearly, we see fluctuations of this historic volatility. Moreover, we see a kind of mean-reversion effect. The peak in the middle of the figures comes from the stock market crash on the 19th of October 1987; windows including this day (with an extremal down-move), give rise to very high volatilities.

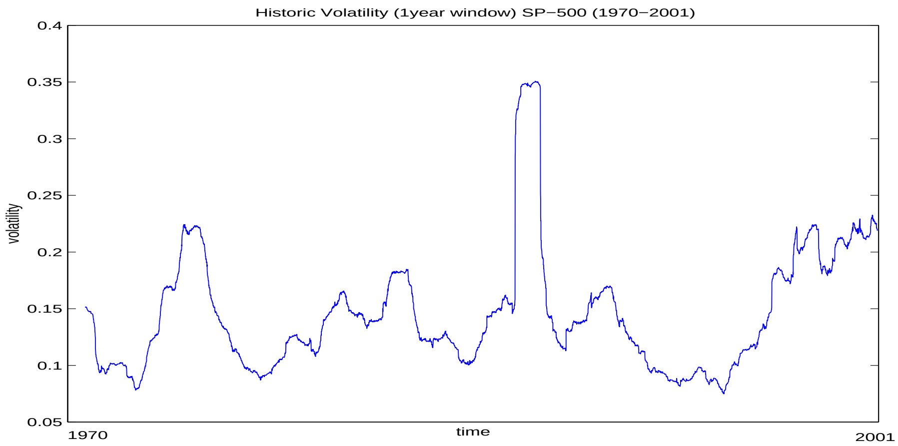

Figure 8: Historic Volatilities on SP-500

2.4.2. Volatility Clusters

Moreover, there is evidence for *volatility clusters*, i.e. there seems to be a succession of periods with high return variance and with low return variance. This can be seen for example in Figure 9, where the absolute log-returns of the SP500-index over a period of more than 30 years is plotted. One clearly sees that there are periods with high absolute log-returns and periods with lower absolute log-returns. This is in contrast with the picture in Figure 10, where similarly the absolute value of simulated normal random variables (with the empirical standard deviation of the SP500) are graphed. Here one sees a more homogeneous picture, often referred to as white noise. Large price variations are more likely to be followed by large price variations. These observations

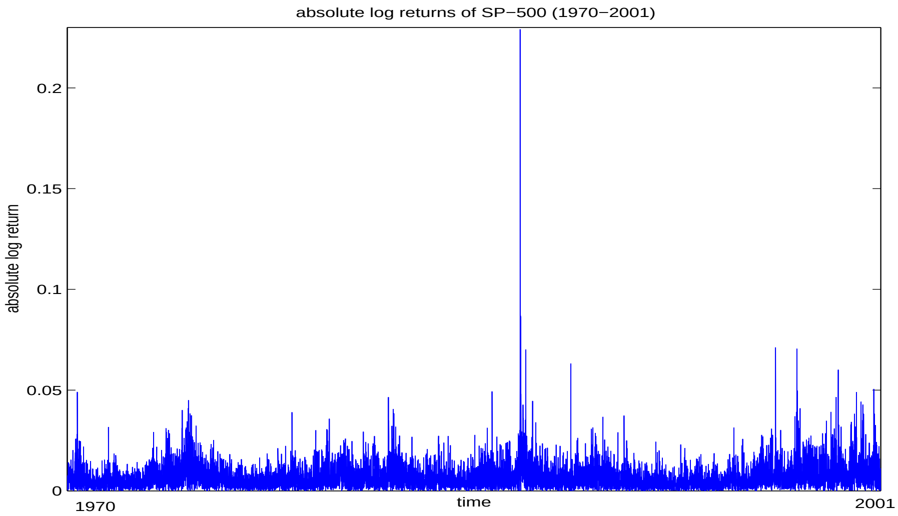

Figure 9: Volatility clusters: absolute log-returns SP500-index between 1970 and 2001

motivate the introduction of models for asset price processes where volatility is itself stochastic.

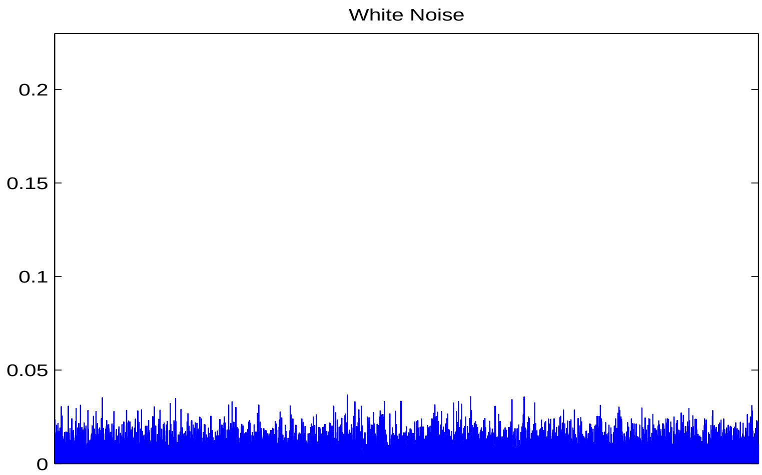

Figure 10: White Noise

2.5. Inconsistency with Market Option Prices

2.5.1. Calibration on Market Prices

If we estimate the model parameters by minimizing the root mean square error between market prices and the Black-Scholes model prices, we can observe an enormous difference. This can be seen in Figure 11 for the SP500-index options. The volatility parameter which gives the best fit in the least-squared sense for the Black-Scholes model is $\sigma = 0.1812$ (in terms of years). Recall that the o-signs are market prices; the +-signs are the calibrated model prices.

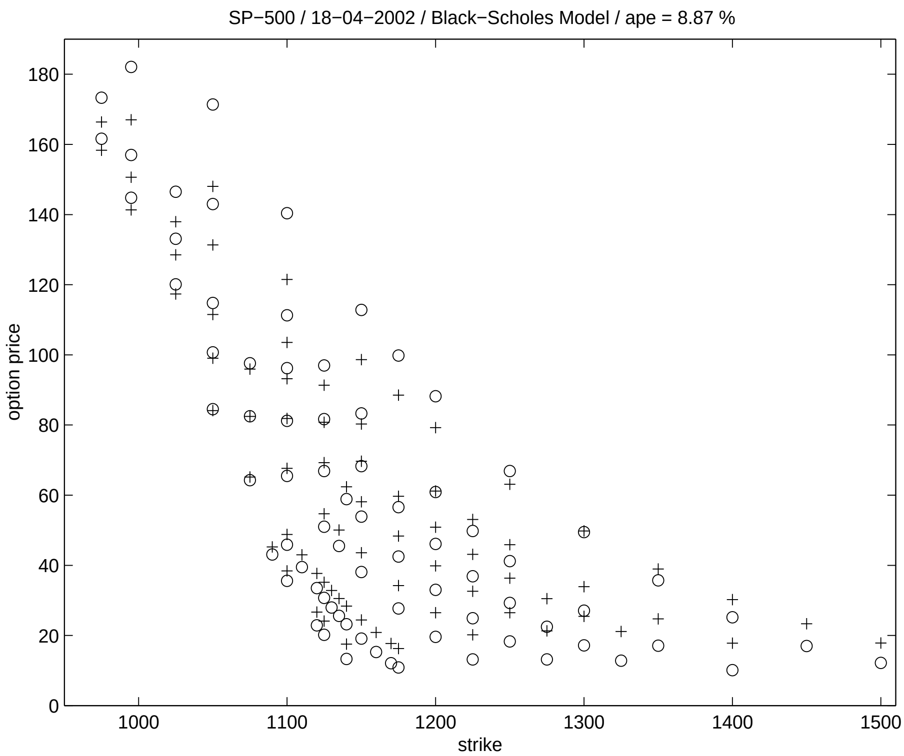

Figure 11: Black Scholes ($\sigma = 0.1812$) calibration on SP500 options (o's are market prices, +'s are model prices)

In Table 4 we give the relevant measures of fit, we introduced in Chapter 1.

Model	**ape**	**aae**	**rmse**	**arpe**
Black-Scholes	8.87 %	5.4868	6.7335	16.92 %

Table 4: *ape*, *aae* and *rmse* of Black-Scholes model calibration on market option prices

2.5.2. Implied Volatility

Another way to see that the classical Black-Scholes model does not correspond with option prices in the market, is by looking at the implied volatilities coming from the option prices. For every European call option with strike K and time to maturity T, we calculate the only (free) parameter involved, the volatility $\sigma = \sigma(K, T)$, such that the theoretical option price (under the Black-Scholes model) matches the empirical one. This $\sigma = \sigma(K, T)$ is called the *implied volatility* of the option. Implied volatility is a timely measure - it reflects the market's perceptions today.

There is no closed formula to extract the implied volatility out of the call option price. We have to rely on numerical methods. One method to find numerically implied volatilities is the classical

Newton-Raphson iteration procedure. Denote by $C(\sigma)$ the price of the relevant call option as a function of volatility. If C is the market price of this option we need to solve the transcendental equation

$$C = C(\sigma) \tag{9}$$

for σ. We start with some initial value we propose for σ; we denote this starting value with σ_0. In terms of years, it turns out that a σ_0 around 0.20 performs very well for most common stocks and indices. In general, if we denote by σ_n the value obtained after n iteration steps, the next value σ_{n+1} is given by

$$\sigma_{n+1} = \sigma_n - \frac{C(\sigma_n) - C}{C'(\sigma_n)},$$

where in the denominator C' refers to the differential with respect to σ of the call price function (this quantity is also referred to as the vega). For the European call option (under Black-Scholes) we have:

$$C'(\sigma_n) = S_0\sqrt{T}\mathrm{N}(d_1) = S_0\sqrt{T}\mathrm{N}\left(\frac{\log(S_0/K) + (r - q + \frac{\sigma_n^2}{2})T}{\sigma_n\sqrt{T}}\right),$$

where S_0 is the current stock price, d_1 as in (6) and $\mathrm{N}(x)$ is the cumulative probability distribution of a Normal$(0,1)$ random variable as in (1).

Next, we bring together for every maturity and strike this volatility σ in Figure 12, where one sees the so-called volatility surface. Under the Black-Scholes model, all σ's should be the same; clearly we observe that there is a huge variation in this volatility parameter both in strike as in time to maturity. One says often there is a volatility smile or skew effect. Again this points to the fact that the Black-Scholes model is not appropriate and the traders already count in this deficiency into their prices.

2.5.3. Implied Volatility Models

Great care has to be taken by using implied volatilities to price options. Fundamentally, using implied volatilities is wrong. Taking different volatilities for different options on the same underlying asset, give rise to different stochastic models for one asset. Moreover, the situation worsens in case of exotic options. [116] showed that if one tries to find the implied volatilities coming out of exotic options like barrier options (see Chapter 9), there are cases where there are two or even three solutions to the implied volatility equation (for the European call option, see Equation (9)). Implied volatilities are thus not unique in these situations. More extremely, if we consider an up-and-out put barrier option, where the strike coincides with the barrier and the risk-free rate equals the dividend yield, the Black-Scholes price (for which there is a formula in closed form available) is independent of the volatility. So if the market price happens to coincide with the computed value, you can have any implied volatility you want. Otherwise there is no implied volatility.

From this, it should be clear that great caution has to be taken by using European call option implied volatilities for exotic options with apparently similar characteristics (like the same strike price for example). There is no guarantee that the obtained prices are reflecting true prices.

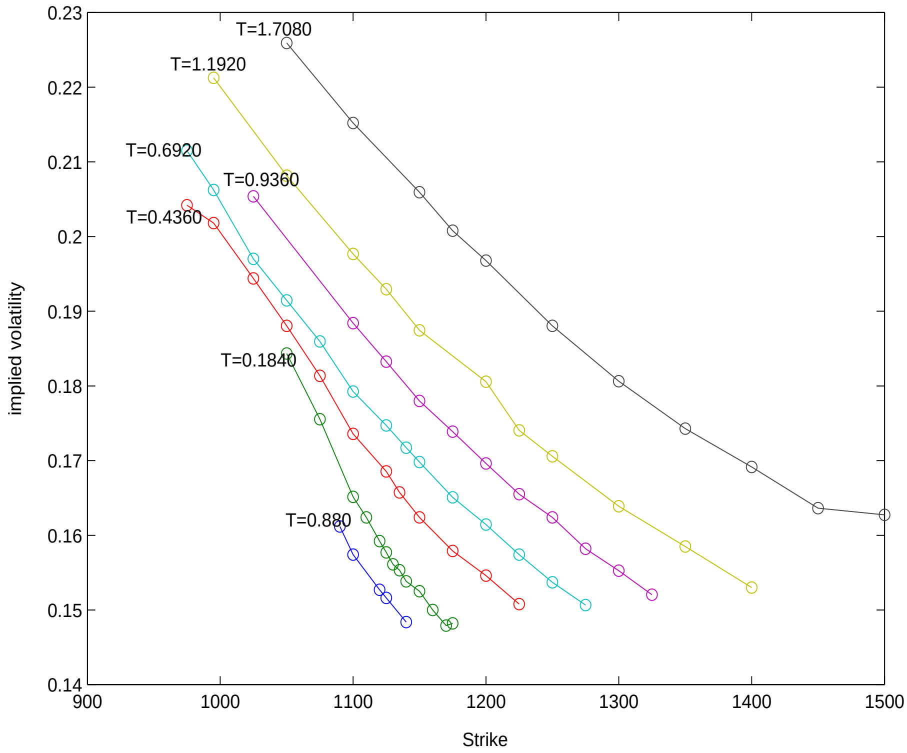

Figure 12: Implied Volatilities

3. THE VG MODEL

In the previous chapter, we have seen that the Black-Scholes model has many imperfections. Which stylized features would one like to have ? As indicated by analyzing empirical data, the following features are under our focus:

- We should have a flexible underlying distribution for log-returns incorporating the possibility of skewness and excess kurtosis.
- Related to this, we would like that the distribution produces more realistic extreme event probabilities; the tails of the distribution should be (at least) semi-heavy tails.
- The model should allow for jumps in the sample paths.
- Stochastic volatility should be possible to incorporate.

On top of that we would like to still have a tractable model. The application of the model in practice stands or falls with its tractability. The calculation of (exotic) option prices and hedge parameters, the generation of sample paths, the calibration of the model etc. should be possible in a reasonable amount of time such that the result is not outdated before it is produced. More precisely, we will focus on models for which

- very fast pricing of European vanillas is possible;

- calibration of the model on a given implied volatility surface can be performed in a reasonable amount of time;
- fast Monte-Carlo simulation is possible in order to do option pricing of exotic options of European type;
- finite-difference or other techniques are available to do pricing of American or barrier products.

Next, we will start our quest with looking for a more flexible distribution.

3.1. The VG distribution

The Gamma distribution will be an essential building block of the construction of the Variance Gamma (VG) distribution on which we will focus a lot on throughout these notes. We start with is the definition and some properties of the Gamma distribution.

3.1.1. The Gamma Distribution

The Gamma distribution is a distribution that lives on the positive real numbers and dependents on two parameters. More precisely, the density function of the Gamma distribution Gamma(a, b) with parameters $a > 0$ and $b > 0$ is given by

$$f_{Gamma}(x; a, b) = \frac{b^a}{\Gamma(a)} x^{a-1} \exp(-xb), \quad x > 0.$$

The density function clearly has a semi-heavy (right) tail; for different parameter values the density function is graphed in Figure 13.

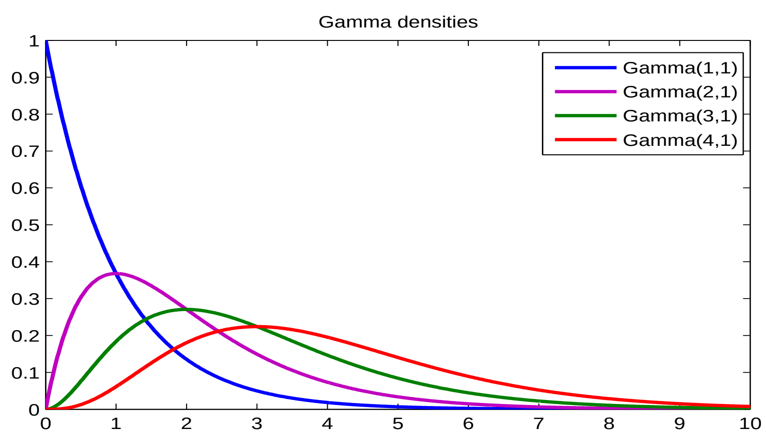

Figure 13: The Gamma density

The characteristic function is given by

$$\phi_{Gamma}(u; a, b) = (1 - \mathrm{i}u/b)^{-a}.$$

The following properties of the Gamma(a, b) distribution can easily be derived from the characteristic function:

	Gamma(a, b)
mean	a/b
variance	a/b^2
skewness	$2a^{-1/2}$
kurtosis	$3(1 + 2a^{-1})$

Note, also that we have the following scaling property: If X is Gamma(a, b), then for $c > 0$, cX is Gamma$(a, b/c)$.

3.1.2. The VG distribution

The Variance Gamma VG(C, G, M) distribution on $(-\infty, +\infty)$ can be constructed as the difference of two gamma random variables. Suppose that X is Gamma$(a = C, b = M)$ random variable and that Y is Gamma$(a = C, b = G)$ random variable and that they are independent of each other. Then

$$X - Y \sim \text{VG}(C, G, M).$$

To derive the characteristic function, we start with noting that

$$\phi_X(u) = (1 - \mathrm{i}u/M)^{-C} \text{ and } \phi_Y(u) = (1 - \mathrm{i}u/G)^{-C}.$$

By using the property (10) (see appendix), we have

$$\phi_{-Y}(u) = (1 + \mathrm{i}u/G)^{-C}.$$

Summing the two independent random variables X and $-Y$ and using the convolution property (11) from the appendix gives

$$\phi_{X-Y}(u) = (1 - \mathrm{i}u/M)^{-C}(1 + \mathrm{i}u/G)^{-C} = \left(\frac{GM}{GM + (M - G)\mathrm{i}u + u^2}\right)^C.$$

Another way of introducing the Variance Gamma (VG) distribution is by mixing a Normal distribution with a Gamma random variate. The procedure goes as follows: Take a random variate $G \sim$ Gamma$(a = 1/\nu, b = 1/\nu)$. Then sample a random variate $X \sim$ Normal$(\theta G, \sigma^2 G)$, then X follows a Variance Gamma distribution. The distribution of X is denoted VG(σ, ν, θ) and thus depends on 3 parameters:

- a real number θ (in the mean of the Normal distribution)
- a positive number σ (in the variance of the Normal distribution)
- a positive number ν (of the Gamma random variable G)

	VG(σ,ν,θ)	**VG**$(\sigma,\nu,0)$
mean	θ	0
variance	$\sigma^2+\nu\theta^2$	σ^2
skewness	$\theta\nu(3\sigma^2+2\nu\theta^2)/(\sigma^2+\nu\theta^2)^{3/2}$	0
kurtosis	$3(1+2\nu-\nu\sigma^4(\sigma^2+\nu\theta^2)^{-2})$	$3(1+\nu)$

Table 5: VG distribution characteristics in the (σ,ν,θ) parametrization.

One can show using basic probabilistic techniques that under this parameter setting the characteristic function of the VG(σ,ν,θ) law is given by

$$E[\exp(\mathrm{i}uX)] = \phi_{VG}(u;\sigma,\nu,\theta) = (1-\mathrm{i}u\theta\nu+\sigma^2\nu u^2/2)^{-1/\nu}.$$

Using elementary calculus one can find the correspondence between the two possible parameter settings. On one hand, we could go from the (σ,ν,θ) setting to the parametrization in terms of $C(arr)$, $G(eman)$ and $M(adan)$ using

$$\begin{aligned} C &= 1/\nu > 0 \\ G &= \left(\sqrt{\frac{\theta^2\nu^2}{4}+\frac{\sigma^2\nu}{2}}-\frac{\theta\nu}{2}\right)^{-1} > 0 \\ M &= \left(\sqrt{\frac{\theta^2\nu^2}{4}+\frac{\sigma^2\nu}{2}}+\frac{\theta\nu}{2}\right)^{-1} > 0. \end{aligned}$$

Going the other way around one can use:

$$\begin{aligned} \nu &= 1/C \\ \sigma^2 &= 2C/(MG) \\ \theta &= C(G-M)/(MG). \end{aligned}$$

Its density function is given by

$$\begin{aligned} f_{\mathrm{VG}}(x;C,G,M)(x) &= \frac{(G\,M)^C}{\sqrt{\pi}\,\Gamma(C)}\exp\left(\frac{(G-M)x}{2}\right) \\ &\quad\times\left(\frac{|x|}{G+M}\right)^{C-1/2} K_{C-1/2}\big((G+M)\,|x|/2\big), \end{aligned}$$

where $\mathrm{K}_\nu(x)$ denotes the modified Bessel function of the third kind with index ν and $\Gamma(x)$ denotes the gamma function.

As shown in Figures 14, 15 and 16 one can see that the distribution is very flexible.

Some distribution characteristics are summarized in the Tables 5 and 6.

When $\theta=0$ the distribution is symmetric. Negative values of θ result in negative skewness; positive θ's give positive skewness. The parameter ν primarily controls the kurtosis.

In terms of the (C,G,M)-parameters this reads as follows:

Under this setting, $G=M$ gives the symmetric case, $G<M$ results in negative skewness and $G>M$ give rise to positive skewness. The parameter C controls the kurtosis.

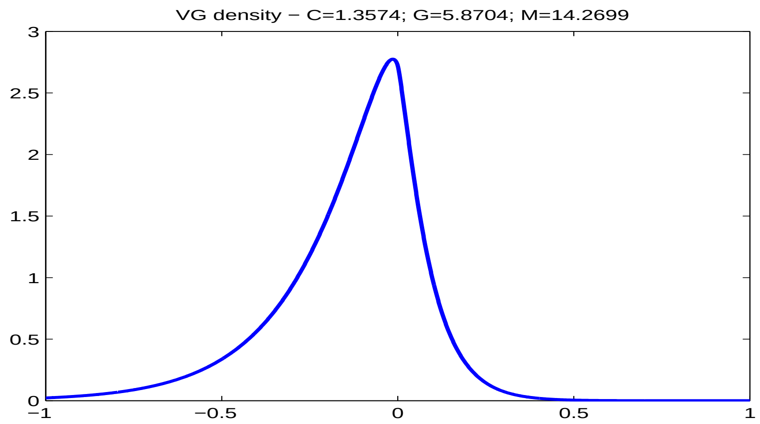

Figure 14: The VG density

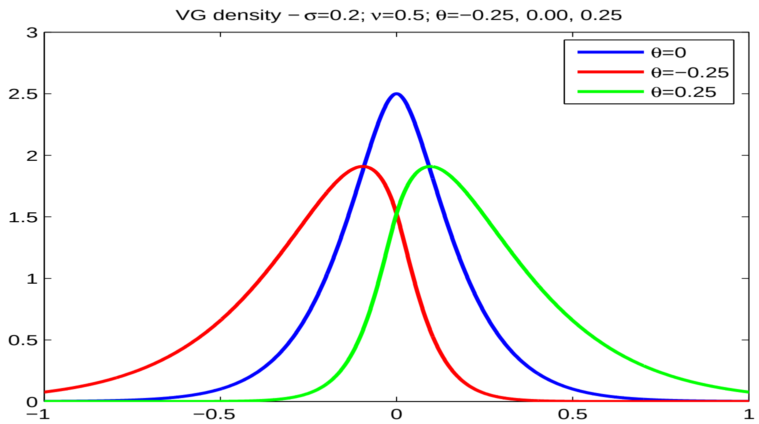

Figure 15: The VG density

If we fit the VG density to the Kernel density, we obtain a very good fit (compare with Normal Fit). In Figure 17, one sees a fit on a data set of daily log-returns of the SP500 over more than 30 years. Statistical χ^2-tests confirm the goodness of fit.

3.2. The VG Process

Recall the definition of a standard Brownian Motion $W = \{W_t, t \geq 0\}$

- W starts at zero: $W_0 = 0$.
- W has independent increments: the distribution of increments over non-overlapping time intervals are stochastically independent.
- W has stationary increments: the distribution of an increment over a time-interval depends only on the length of the interval; not on the exact location.

	VG(C,G,M)	**VG**(C,G,G)
mean	$C(G-M)/(MG)$	0
variance	$C(G^2+M^2)/(MG)^2$	$2CG^{-2}$
skewness	$2C^{-1/2}(G^3-M^3)/(G^2+M^2)^{3/2}$	0
kurtosis	$3(1+2C^{-1}(G^4+M^4)/(M^2+G^2)^2)$	$3(1+C^{-1})$

Table 6: VG distribution characteristics in the (C,G,M) parametrization.

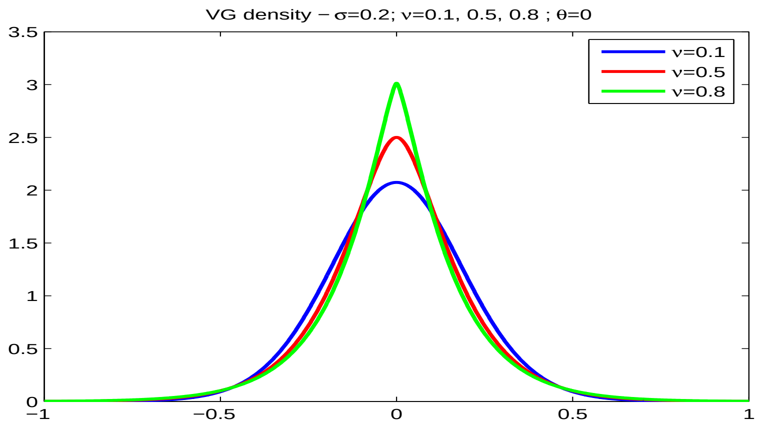

Figure 16: The VG density

- $W_{s+t}-W_t \sim$ Normal$(0,s)$: increments are Normally distributed.

One can define in a similar way a stochastic process based on the VG distribution. (For mathematical details and other examples see [113]). A stochastic process $X=\{X_t, t\geq 0\}$ is a Variance-Gamma Process with parameters C,G,M if

- X starts at zero: $X_0=0$.
- X has independent increments.
- X has stationary increments.
- Furthermore we have that $X_{s+t}-X_t \sim$ VG(Cs,G,M), i.e. increments are VG distributed;

It will turn out (see again [113]) that a VG process is a pure jump process. Sample paths have no diffusion component in contrast with a Brownian motion (see Figure 18).

3.3. The VG Stock Price Model

Instead of modeling the stock price process as an exponential of a Brownian Motion (with drift):

$$S_t = S_0 \exp((\mu-\sigma^2/2)t+\sigma W_t), \qquad S_0>0,$$

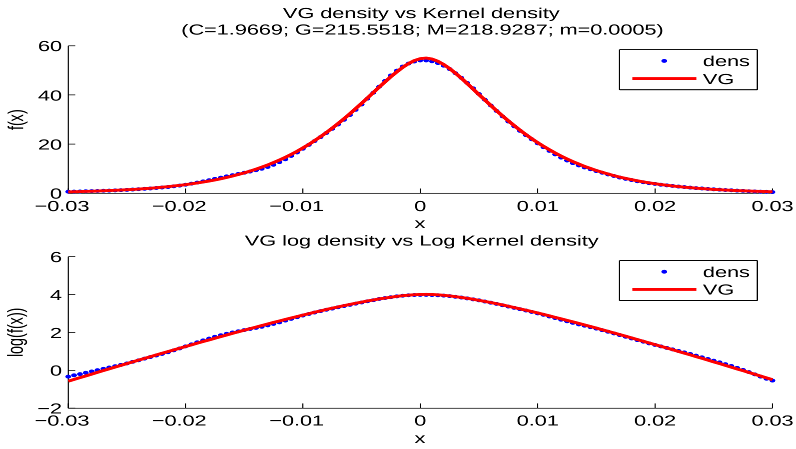

Figure 17: fitting the VG density to the empirical Kernel density of SP500 data.

we now model S as the exponential of a VG process $X = \{X_t, t \geq 0\}$:

$$S_t = S_0 \exp(X_t), \qquad S_0 > 0.$$

In that way, log-returns no longer are Normally distributed but follow the more flexible VG distribution:

$$\log S_{t+1} - \log S_t = X_{t+1} - X_t \sim \text{VG}(C, G, M), \qquad C, G, M > 0.$$

Note that under Black-Scholes we had:

$$\log S_{t+1} - \log S_t \sim \text{Normal}\left(\mu - \frac{\sigma^2}{2}, \sigma^2\right).$$

Under a Black-Scholes framework moving from a historical world to a risk-neutral one is easy: one replaces the drift μ with the interest rate r (minus the dividend yield q).

$$S_t = S_0 \exp((r - q - \sigma^2/2)t + \sigma W_t), t \geq 0,$$

In contrast with the BS-world; for the VG model (and in general for all more advanced models), there is no unique transformation. Actually, there are infinitely many possible measure changes. On particular easy transformation is the mean-correcting measure change, where the VG process is shifted in order to obtain a martingale.

$$S_t = S_0 \exp((r - q + \omega)t + X_t), t \geq 0,$$

where

$$\omega = \nu^{-1} \log\left(1 - \frac{1}{2}\sigma^2\nu - \theta\nu\right)$$

Note that, most of the time we immediately will work under a risk-neutral setting (after calibrating the model to market data) and we do not have to worry about the measure change.

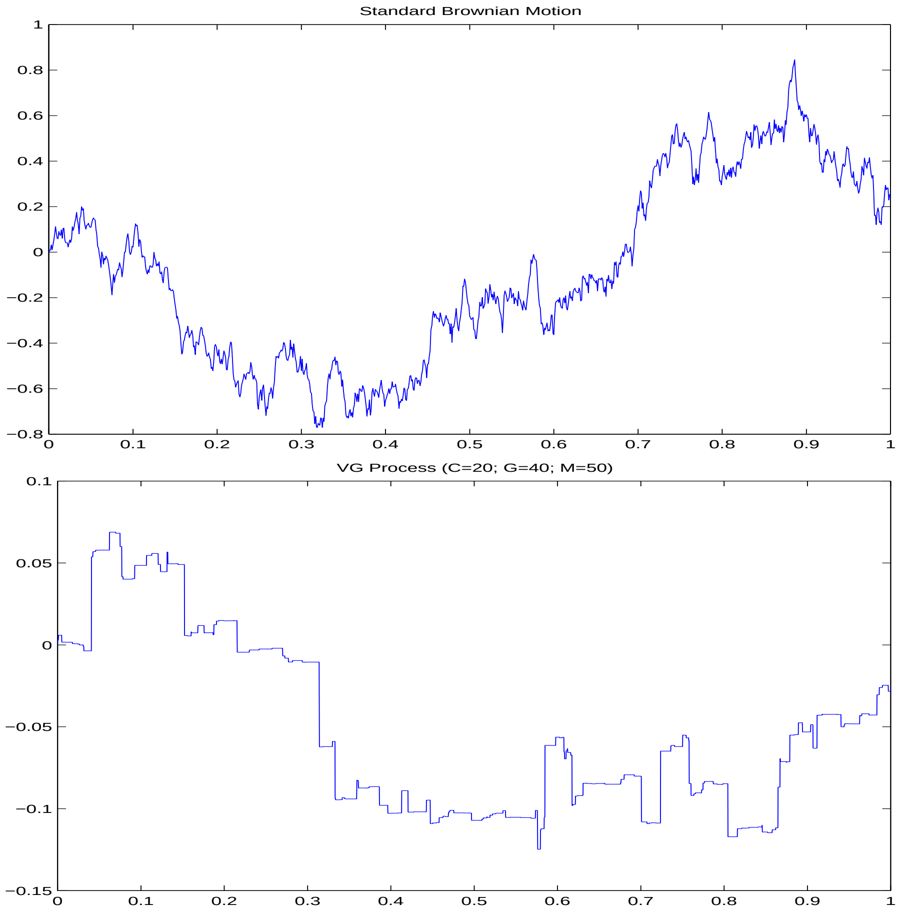

Figure 18: Brownian Motion and VG paths

4. PRICING VANILLAS USING FFT

In this Chapter, we describe how one can price very fast and efficiently vanilla options using the theory of characteristic functions and Fast Fourier Transforms. Our aim is to develop a solid understanding of the current frameworks for pricing of vanilla derivatives using these techniques and to give readers the mathematical and practical background necessary to apply and implement the techniques. The method is particularly interesting in case of advanced equity models, like Variance Gamma model, its stochastic volatility extension, and many other models like the Heston model, where no closed-form solutions for vanillas exist.

An important advantage of the method is that the pricer only needs as input the characteristic function of the dynamics of the underlying model. If one likes to switch to another model, only the corresponding characteristic functions needs to be changed and the actual pricing algorithm remains untouched. The methodology can not only be applied to vanillas, but typically to more general options which depend only on the stock price at maturity.

Furthermore, a lot of the greeks of the vanilla can also be calculated using a similar procedure.

4.1. Pricing of European Call Options using Characteristic Functions

4.1.1. THE CARR-MADAN FORMULA

According to the fundamental theorem of asset pricing the arbitrage free price $C(K,T)$ of an European call option with maturity T and strike K is given by

$$C(K,T) = \exp(-rT)E_Q[(S_T - K)^+],$$

where we take the expectation under Q, i.e. the risk-neutral martingale measure. If we have the density function available (as in the VG case), we could in principle calculate:

$$C(K,T) = \exp(-rT)\int_{-\infty}^{+\infty} f_{VG}(x;CT,G,M)\left(S_0\exp((r-q+\omega)T+x)-K\right)^+ \mathrm{d}x$$

But this is typically time-consuming and not that trivial (Bessel function!), moreover in many other situations like in many models incorporating stochastic volatility (e.g. Heston or VG with stochastic vol) no density function is available in closed-form.

Much faster is the application of the Carr-Madan formula:

$$C(K,T) = \frac{\exp(-\alpha\log(K))}{\pi}\int_0^{+\infty}\exp(-\mathrm{i}v\log(K))\varrho(v)\mathrm{d}v,$$

where

$$\varrho(v) = \frac{\exp(-rT)E[\exp(\mathrm{i}(v-(\alpha+1)\mathrm{i})\log(S_T))]}{\alpha^2+\alpha-v^2+\mathrm{i}(2\alpha+1)v}$$

Important in the formulas is the (risk neutral - under Q say) characteristic function of the log price process $s_T = \log(S_T)$ at maturity T.

$$\phi(u;T) = E_Q[\exp(\mathrm{i}u\log(S_T))] = E_Q[\exp(\mathrm{i}us_T)]$$

In many situation $\phi(u;T)$ is known analytically: BS, VG, Lévy Models, Heston, Heston with jumps, Lévy models with stochastic vol, ... In that case $\varrho(v)$ can be expressed completely analytically. Indeed the expected value, $E[\exp(\mathrm{i}(v-(\alpha+1)\mathrm{i})\log(S_T))]$, in $\varrho(v)$ is nothing else that the characteristic function of the random variable $\log(S_T)$ evaluated in the point $v-(\alpha+1)\mathrm{i}$:

$$E[\exp(\mathrm{i}(v-(\alpha+1)\mathrm{i})\log(S_T))] = \phi(v-(\alpha+1)\mathrm{i};T),$$

Example: In Black-Scholes world

$$\begin{aligned}
S_T &= S_0\exp((r-q-\sigma^2/2)T+\sigma W_T), \text{ with } W \text{ standard Brownian motion}\\
s_T &= \log(S_0)+(r-q-\sigma^2/2)T+\sigma W_T\\
s_T &\sim \text{Normal}(\log(S_0)+(r-q-\sigma^2/2)T, \sigma^2 T)\\
\phi_{BS}(u;T) &= \exp(\mathrm{i}u(\log(S_0)+(r-q-\sigma^2/2)T))\exp\left(-\frac{1}{2}\sigma^2 T u^2\right)
\end{aligned}$$

Example: In the VG world

$$\begin{aligned} S_T &= S_0 \exp((r-q+\omega)T + X_T), \text{ with } X \text{ is a VG}(C,G,M) \text{ process} \\ s_T &= \log(S_0) + (r-q+\omega)T + X_T \\ s_T &\sim \log(S_0) + (r-q+\omega)T + \text{VG}(CT,G,M) \\ \phi_{VG}(u;T) &= \exp(\mathrm{i}u(\log(S_0) + (r-q+\omega)T)) \left(\frac{GM}{GM + (M-G)\mathrm{i}u + u^2} \right)^{CT} \end{aligned}$$

Using this formula and combining it with numerical techniques that evaluate the integral involved in a efficient way (based on the Fast Fourier Transform (FFT)), will lead to an extreme fast pricing algorithm of the entire option surface. The algorithm generates in one run prices for a fine grid of strikes and all given maturities. Moreover the formula/algorithm is generic and can be used for any model if the characteristic $\phi(u;T)$ is available. We will illustrate that using these pricing formula/algorithm, very fast global calibration on market option data for advanced models is possible.

Basically we provide the following input (see Figure 19) to our pricing algorithm:

- characteristic function of underlying stochastic dynamics;
- parameters;
- maturities.

The algorithm generates as output:

- for a whole range of strikes (chosen by FFT algorithm) and all the given maturities: vanilla prices or equivalently the implied volatilities.

If one needs the option price for a particular strike, this is obtain via interpolation. The strike-grid should hence be taken fine enough to give accurate results.

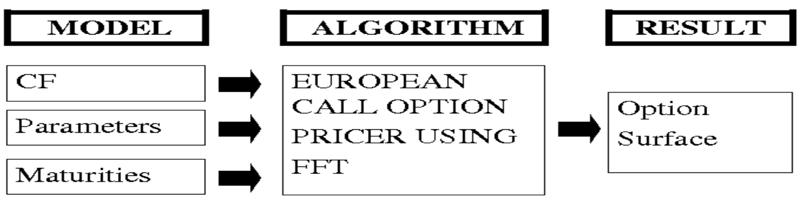

Figure 19: Pricing of vanillas using CF - Algorithm I/O

Next, we will prove the Carr-Madan formula and we will show how the Fast Fourier Transform (FFT) can be used to evaluate the integral in the formula. Let α be a positive constant such that the αth moment of the stock price exists. We comment later on the choice of α. Recall, that we suppose that we have explicitly available the characteristic function of $s_T = \log(S_T)$:

$$\phi(u;T) = E_Q[\exp(\mathrm{i}us_T)] = E_Q[\exp(\mathrm{i}u\log(S_T)].$$

Denote by $k = \log(K)$ the log-strike, so as we moved from S_T to s_T, we also move to log-space for the strikes. If we work in log space we also denote the call price with log-strike k with $C(k, T)$.

Assume for simplicity that the density function of $s_T = \log(S_T)$ exists and let us denote this density function by $q(x; T)$. Then we have:

$$\phi(u, T) = \int_{-\infty}^{+\infty} \exp(\mathrm{i}ux) q(x; T) \mathrm{d}x.$$

We know

$$\begin{aligned} C(k, T) &= \exp(-rT) E_Q[(S_T - \mathrm{e}^k)^+] \\ &= \exp(-rT) \int_k^{\infty} (\mathrm{e}^x - \mathrm{e}^k) q(x; T) \mathrm{d}x. \end{aligned}$$

However, note that the Call function $C(k, T) \to S_0$ (in the log-strike price) as $k \to -\infty$ and is hence not square integrable and it would not be possible to apply Fourier theory. To obtain a square integrable function, we consider the modified call price:

$$c(k; T) = \exp(\alpha k) C(k; T),$$

for some $\alpha > 0$. For a suitable range of positive values for α, we expect $c(k, T)$ to be square integrable in k over the entire real line.

It will turn out that $\varrho(v)$ is the Fourier transform of $c(k; T)$. Indeed

$$\begin{aligned} &\int_{-\infty}^{+\infty} \exp(\mathrm{i}vk) c(k; T) \mathrm{d}k \\ &= \int_{-\infty}^{+\infty} \exp(\mathrm{i}vk) \exp(\alpha k) C(k; T) \mathrm{d}k \\ &= \int_{-\infty}^{+\infty} \exp(\mathrm{i}vk) \exp(-rT) \exp(\alpha k) \int_k^{\infty} (\mathrm{e}^x - \mathrm{e}^k) q(x; T) \mathrm{d}x \mathrm{d}k \\ &= \exp(-rT) \int_{-\infty}^{+\infty} q(x; T) \int_{-\infty}^{x} \exp(\mathrm{i}vk) \exp(\alpha k) (\mathrm{e}^x - \mathrm{e}^k) \mathrm{d}k \mathrm{d}x \\ &= \exp(-rT) \int_{-\infty}^{+\infty} q(x; T) \left(\frac{\exp\left((\alpha + 1 + \mathrm{i}v)x\right)}{\alpha^2 + \alpha - v^2 + \mathrm{i}(2\alpha + 1)v} \right) \mathrm{d}x \\ &= \frac{\exp(-rT) \phi(v - (\alpha + 1)\mathrm{i}, T)}{\alpha^2 + \alpha - v^2 + \mathrm{i}(2\alpha + 1)v} \\ &= \varrho(v). \end{aligned}$$

Then using the inverse transform we have

$$\begin{aligned} C(k, T) &= \exp(-\alpha k) c(k; T) \\ &= \exp(-\alpha k) \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-\mathrm{i}vk) \varrho(v) \mathrm{d}v \\ &= \exp(-\alpha k) \frac{1}{\pi} \int_0^{\infty} \exp(-\mathrm{i}vk) \varrho(v) \mathrm{d}v. \end{aligned}$$

The second equality holds because $C(k,T)$ is real, which implies that the function $\varrho(v)$ is odd in its imaginary part and even in its real part.

Rephrasing gives the Carr-Madan formula:

$$C(K,T) = \frac{\exp(-\alpha \log(K))}{\pi} \int_0^{+\infty} \exp(-\mathrm{i}v \log(K))\varrho(v)\mathrm{d}v,$$

where

$$\begin{aligned}\varrho(v) &= \frac{\exp(-rT)E_Q[\exp(\mathrm{i}(v-(\alpha+1)\mathrm{i})\log(S_T))]}{\alpha^2+\alpha-v^2+\mathrm{i}(2\alpha+1)v} \\ &= \frac{\exp(-rT)\phi(v-(\alpha+1)\mathrm{i};T)}{\alpha^2+\alpha-v^2+\mathrm{i}(2\alpha+1)v}.\end{aligned}$$

Next, we illustrate how the calculation of the call price via the Carr-Madan formula can be done fast and accurately using the Fast Fourier Transform (FFT). FFT is an efficient algorithm for computing the following transformation of a vector $(\alpha_n, n = 1,\dots,N)$ into a vector $(\beta_n, n = 1,\dots,N)$:

$$\beta_n = \sum_{j=1}^{N} \exp\left(-\frac{\mathrm{i}2\pi(j-1)(n-1)}{N}\right)\alpha_j.$$

Typically N is a power of 2. The number of operations of the FFT algorithm is of the order $\mathcal{O}(N\log N)$ and this in contrast to the straightforward evaluation of the above sums which give rise to $\mathcal{O}(N^2)$ numbers of operations.

An approximation for the integral in the Carr-Madan formula

$$C(k,T) = \exp(-\alpha k)\frac{1}{\pi}\int_0^{\infty} \exp(-\mathrm{i}vk)\varrho(v)\mathrm{d}v$$

on the N points-grid $(0, \eta, 2\eta, 3\eta, ..., (N-1)\eta)$ is

$$C(k,T) \approx \exp(-\alpha k)\frac{1}{\pi}\sum_{j=1}^{N}\exp(-\mathrm{i}v_j k)\varrho(v_j)\eta, \qquad v_j = \eta(j-1).$$

We will calculate the value of these call prices for N log-strikes levels ranging from say $-b$ to b (Note: if $S_0 = 1$, at-the-money corresponds to $b = 0$):

$$k_n = -b + \lambda(n-1), \qquad n = 1,\dots,N, \quad \text{where } \lambda = 2b/N.$$

This gives

$$\begin{aligned}C(k_n,T) &\approx \exp(-\alpha k_n)\frac{1}{\pi}\sum_{j=1}^{N}\exp(-\mathrm{i}v_j(-b+\lambda(n-1)))\varrho(v_j)\eta, \\ &= \exp(-\alpha k_n)\frac{1}{\pi}\sum_{j=1}^{N}\exp(-\mathrm{i}\eta\lambda(j-1)(n-1))\exp(\mathrm{i}v_j b)\varrho(v_j)\eta.\end{aligned}$$

If we choose λ and η such that $\lambda\eta = 2\pi/N$, then

$$C(k_n, T) \approx \exp(-\alpha k_n)\frac{1}{\pi}\sum_{j=1}^{N}\exp\left(-\frac{\mathrm{i}2\pi(j-1)(n-1)}{N}\right)\exp(\mathrm{i}v_j b)\varrho(v_j)\eta.$$

The summation above is an exact application of the FFT on the vector $(\exp(\mathrm{i}v_j b)\varrho(v_j)\eta, j = 1,\ldots,N)$. Note that by fixing $\lambda\eta = 2\pi/N$, taking a smaller grid-size η makes the grid-size λ (for the log-strike grid) larger. Carr and Madan (1999) report that the following choice gave very satisfactory results:

$$\begin{aligned} \eta &= 0.25 \\ N &= 4096 \\ \alpha &= 1.5 \end{aligned}$$

which implies

$$\begin{aligned} \lambda &= 0.0061 \text{ or an interstrike range a little over a half a percentage} \\ b &= 12.57 \end{aligned}$$

A more refined weighting (Simpson's rule) for the integral in the Carr-Madan formula on the N points-grid $(0, \eta, 2\eta, 3\eta, ..., (N-1)\eta)$ leads to the following approximation

$$C(k, T) \approx \exp(-\alpha k)\frac{1}{\pi}\sum_{j=1}^{N}\exp(-\mathrm{i}v_j k)\varrho(v_j)\eta\left(\frac{3+(-1)^j-\delta_{j-1}}{3}\right), \qquad v_j = \eta(j-1)$$

and gives a more accurate integration.

4.2. Fast Calibration on vanillas

A calibration procedure looks (see Figure 20) for the optimal parameter set such that model prices match as best as possible the market prices.

For performing a calibration, we provide the following input to our calibration algorithm:

- characteristic function of underlying stochastic dynamics;
- initial guess of parameters;
- market prices.

By calling many times the pricing algorithm (for the maturities available in the data) the optimalization procedure searches the parameter space (starting at the initial guess) and minimizes the error between model prices and the market prices. As output (see Figure 21) one obtains a kind of optimal parameters for which the related model prices fits the market prices best. One of the most widely used direct search methods for nonlinear unconstrained optimization problems is the Nelder-Mead simplex algorithm. Nelder-Mead's algorithm is parsimonious in the number

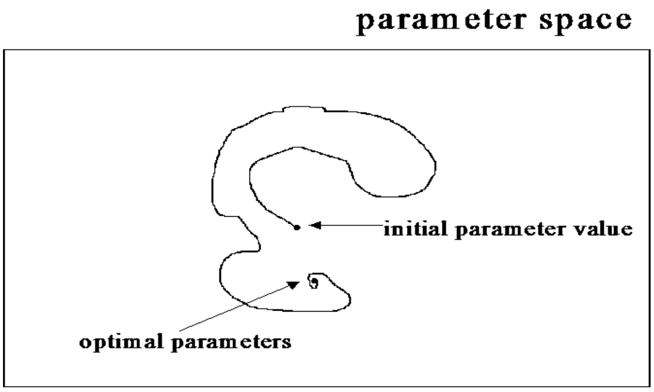

Figure 20: Search Algorithm

of function evaluations per iteration, and is often able to find reasonably good solutions quickly. The algorithm is built in in Matlab and other mathematical software packages. Basically a non-degenerate simplex (a geometric figure in n dimensions of nonzero volume) around the initial guess is created and for the points on the simplex their respective function values (which has to be minimized) is evaluated. In each iteration, typically around the best guess until then, new points are computed, along with their function values, to form a new simplex. The algorithm terminates when the function values at the vertices of the simplex satisfy a predetermined condition.

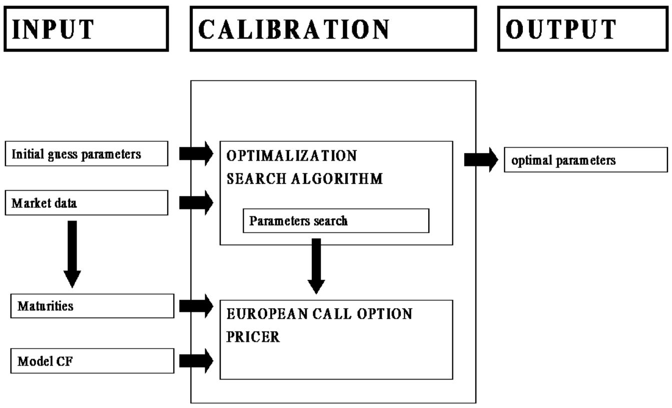

Figure 21: Calibration Algorithm I/O

4.3. The Greeks

To analyze risks involved in a particular option, one often calculates the partial derivatives of the price of the option with respect to its parameters. These partial derivatives are commonly known as *greeks*. Below we detail the calculations for the Delta, the Gamma, the Rho and the Theta. Other greeks (more intrinsically related to the Black-Scholes setting) like Vega, the sensitivity of the option's price with respect to its implied volatility are not available, because there is no relationship between the characteristic function and the variable.

4.3.1. Delta

The *Delta* of an option, often denoted by Δ, measures the sensitivity of the option's value to price changes in the underlying.

$$\Delta = \frac{\partial C(K,T)}{\partial S_0}.$$

In the Black-Scholes setting this Delta is giving the number of stocks one needs to hold in order to perfectly hedge the option. In the more advanced models, perfect hedging is no longer always possible and many other hedging strategies can be considered and maybe make more sense. Nevertheless, Delta-hedging is a very well-accepted strategy that is also straightforward to apply. We must note however that taking (partial) derivatives is a local operator, typically taking into account continuous movements of the underlying. If we work with models in which the underlying price process can jump, such a local operator does not tell the whole story.

We have :

$$\begin{aligned}
\Delta &= \frac{\partial}{\partial S_0}\left[\frac{\exp(-\alpha\log(K))}{\pi}\int_0^{+\infty}\exp(-\mathrm{i}v\log(K))\frac{\exp(-rT)\phi(v-(\alpha+1)\mathrm{i};T)}{\alpha^2+\alpha-v^2+\mathrm{i}(2\alpha+1)v}\mathrm{d}v\right]\\
&= \frac{\exp(-\alpha\log(K))}{\pi}\int_0^{+\infty}\exp(-\mathrm{i}v\log(K))\frac{\exp(-rT)\frac{\partial\phi(v-(\alpha+1)\mathrm{i};T)}{\partial S_0}}{\alpha^2+\alpha-v^2+\mathrm{i}(2\alpha+1)v}\mathrm{d}v.
\end{aligned}$$

Now, assume the (risk-neutral) model for the price of the underlying at time T is of the form $S_T = S_0\bar{S}_T$, where $\bar{S}_T$ is not depending on S_0 anymore, then $\log(S_T) = \log(S_0) + \log(\bar{S}_T)$. Note that this assumption is a very typical one and is the case for all models we consider. Furthermore, we have then that

$$\begin{aligned}
\frac{\partial\phi(v-(\alpha+1)\mathrm{i};T)}{\partial S_0} &= \frac{\partial}{\partial S_0}E[\exp(\mathrm{i}(v-(\alpha+1)\mathrm{i})(\log(S_0)+\log(\bar{S}_T)))]\\
&= \frac{\phi(v-(\alpha+1)\mathrm{i};T)(\alpha+1+\mathrm{i}v)}{S_0}.
\end{aligned}$$

In conclusion

$$\begin{aligned}
\Delta &= \frac{\exp(-\alpha\log(K))}{\pi}\int_0^{+\infty}\exp(-\mathrm{i}v\log(K))\frac{\exp(-rT)\phi(v-(\alpha+1)\mathrm{i};T)(\alpha+1+\mathrm{i}v)}{S_0(\alpha^2+\alpha-v^2+\mathrm{i}(2\alpha+1)v)}\mathrm{d}v;\\
&= \frac{\exp(-\alpha\log(K))}{\pi}\int_0^{+\infty}\exp(-\mathrm{i}v\log(K))\frac{\exp(-rT)\phi(v-(\alpha+1)\mathrm{i};T)}{S_0(\alpha+\mathrm{i}v)}\mathrm{d}v,
\end{aligned}$$

where we used in the last line the fact that $\alpha^2+\alpha-v^2+\mathrm{i}(2\alpha+1)v = (\alpha+\mathrm{i}v)(\alpha+1+\mathrm{i}v)$.

4.3.2. Gamma

The *Gamma* of an option, often denoted by Γ, measures the sensitivity of the Delta of the option with respect to price changes in the underlying.

$$\Gamma = \frac{\partial\Delta}{\partial S_0} = \frac{\partial^2 C(K,T)}{\partial S_0^2}.$$

High Gammas mean that the corresponding Delta-hedge position is very sensitive to changes in the underlying price process.

Completely analogous as in the calculation of Delta, we have

$$\Gamma = \frac{\exp(-\alpha\log(K))}{\pi}\int_0^{+\infty}\exp(-\mathrm{i}v\log(K))\frac{\exp(-rT)\frac{\partial^2\phi(v-(\alpha+1)\mathrm{i};T)}{\partial S_0^2}}{\alpha^2+\alpha-v^2+\mathrm{i}(2\alpha+1)v}\mathrm{d}v.$$

Under the same assumption on the form of the price process, we have then that

$$\frac{\partial}{\partial S_0}\frac{\phi(v-(\alpha+1)\mathrm{i};T)}{S_0} = \frac{\phi(v-(\alpha+1)\mathrm{i};T)(\alpha+\mathrm{i}v)}{S_0^2}.$$

Hence, in conclusion

$$\Gamma = \frac{\exp(-\alpha\log(K))}{\pi}\int_0^{+\infty}\exp(-\mathrm{i}v\log(K))\frac{\exp(-rT)\phi(v-(\alpha+1)\mathrm{i};T)}{S_0^2}\mathrm{d}v.$$

4.3.3. Rho

Rho of an option, often denoted by ρ (not to be confused with correlation later on), measures the sensitivity of the option's value with respect to the risk free interest rate r:

$$\rho = \frac{\partial C(K,T)}{\partial r}.$$

If we assume that the (risk-neutral) price of the underlying at time T is of the form $S_T = \exp(rT)\hat{S}_T$, where $\hat{S}_T$ is not depending on r anymore, then $\log(S_T) = rT + \log(\hat{S}_T)$. Note that this assumption is again a very typical one. Calculations are performed in exactly the same was as above; we find

$$\rho = \frac{\exp(-\alpha\log(K))}{\pi}\int_0^{+\infty}\exp(-\mathrm{i}v\log(K))\frac{T\exp(-rT)\phi(v-(\alpha+1)\mathrm{i};T)}{\alpha+1+\mathrm{i}v}\mathrm{d}v.$$

4.3.4. Theta

Also the value of Theta, Θ, i.e. the sensitivity of the option price with respect to the passage of time can be calculated along the same lines. However, here the calculations dependent much more on the explicit form of the characteristic function.

In general we have

$$\begin{aligned}\Theta &= \frac{\partial}{\partial T}\left[\frac{\exp(-\alpha\log(K))}{\pi}\int_0^{+\infty}\exp(-\mathrm{i}v\log(K))\frac{\exp(-rT)\phi(v-(\alpha+1)\mathrm{i};T)}{\alpha^2+\alpha-v^2+\mathrm{i}(2\alpha+1)v}\mathrm{d}v\right]\\ &= \frac{\exp(-\alpha\log(K))}{\pi}\int_0^{+\infty}\exp(-\mathrm{i}v\log(K))\frac{\frac{\partial}{\partial T}[\exp(-rT)\phi(v-(\alpha+1)\mathrm{i};T)]}{\alpha^2+\alpha-v^2+\mathrm{i}(2\alpha+1)v}\mathrm{d}v.\\ &= \frac{\exp(-\alpha\log(K)-rT)}{\pi}\int_0^{+\infty}\exp(-\mathrm{i}v\log(K))\frac{\frac{\partial}{\partial T}\phi(v-(\alpha+1)\mathrm{i};T)-r\phi(v-(\alpha+1)\mathrm{i};T)}{\alpha^2+\alpha-v^2+\mathrm{i}(2\alpha+1)v}\mathrm{d}v.\end{aligned}$$

In case of the VG setting where

$$\phi_{VG}(u;T) = \exp\left(\mathrm{i}u\log(S_0) + T\left(\mathrm{i}u(r-q+\omega) + C\log\left(\frac{GM}{GM+(M-G)\mathrm{i}u+u^2}\right)\right)\right),$$

we have that

$$\frac{\partial}{\partial T}\phi(u;T) = \phi(u;T)\left(\mathrm{i}u(r-q+\omega) + C\log\left(\frac{GM}{GM+(M-G)\mathrm{i}u+u^2}\right)\right).$$

4.4. Calibration Results

Doing a global calibration for the VG model introduced in the previous Chapter, we obtain a serious improvement over the Black-Scholes case. However observe still a significant difference with real market prices as can be seen in Figure 22. The initial parameters we started with where $(C = 1,\ G = 5,\ M = 5)$; the final optimal parameters coming out of the calibration procedure are given by: $(C = 1.3574,\ G = 5.8703,\ M = 14.2699)$

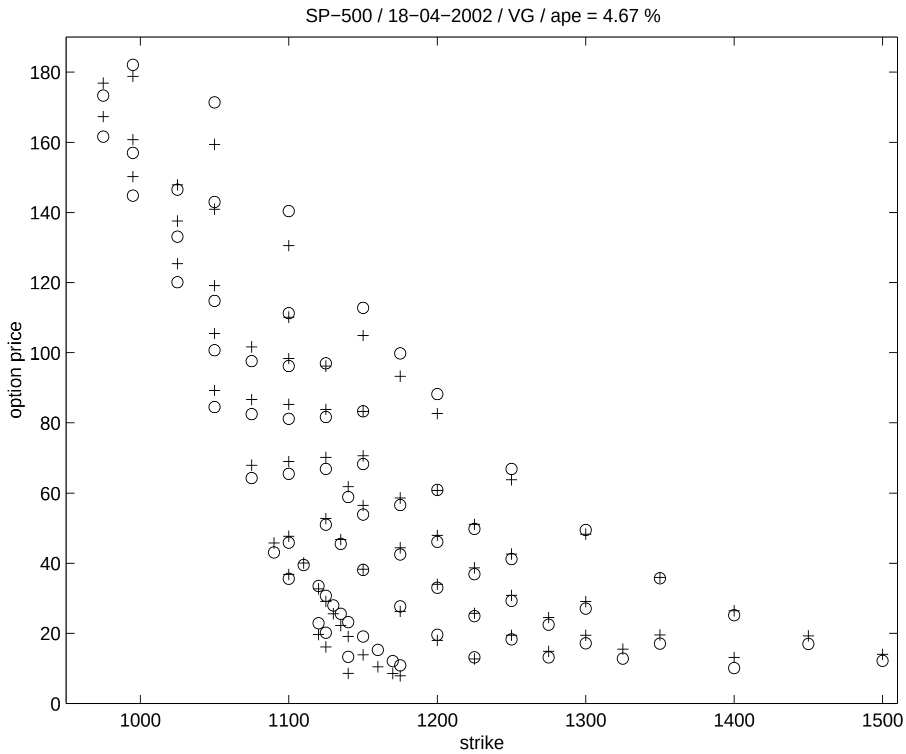

Figure 22: Global Calibration of VG

Fitting the smile at one maturity is done very accurately as can be seen in Figure 23. The parameters coming out of the calibration procedure for each maturity are given in Table 7. We see a quite typical term-structure for the σ and ν parameter.

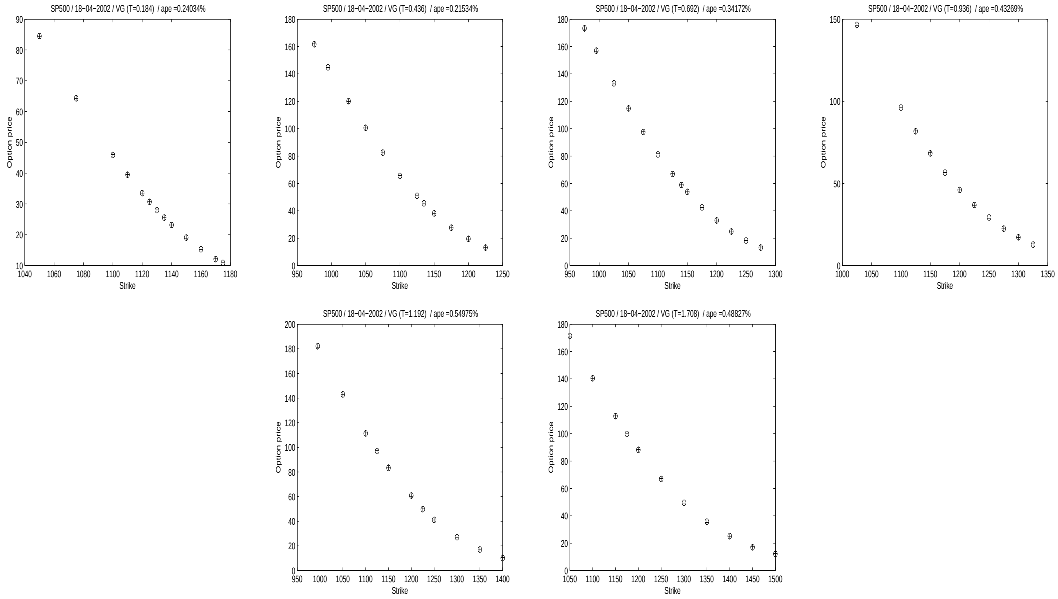

Figure 23: Calibration of VG maturity per maturity

T	σ	ν	θ
0.184	0.1600	0.1141	-0.1788
0.436	0.1631	0.2702	-0.1639
0.692	0.1665	0.4888	-0.1459
0.936	0.1660	0.6263	-0.1545
1.192	0.1720	0.9109	-0.1485
1.708	0.1864	1.5414	-0.1396

Table 7: VG optimal parameters maturity per maturity

5. MONTE CARLO SIMULATION

Next, we look at possible simulation techniques for the processes encountered so far. A Lévy process can be in general simulated based on a compound Poisson approximation. However, typically, for very specific processes like the VG other much faster techniques are available.

We assume we have random number generators at hand which can provide us Normal$(0, 1)$ and Gamma(a, b) random numbers. Throughout $\{v_n\}$ always denotes a Normal$(0, 1)$ random number; $\{g_n\}$ a Gamma random numbers.

A good reference book is [37].

5.1. Brownian Motion

Recall that a standard Brownian motion $W = \{W_t, t \geq 0\}$ has Normal distributed independent increments. We discretize time by taking time steps of size Δt, which we assume to be very small. We simulate (by the Euler scheme) the value of the Brownian motion at the time points

$\{n\Delta t, n = 0, 1, \ldots\}$:

$$W_0 = 0, \quad W_{n\Delta t} = W_{(n-1)\Delta t} + \sqrt{\Delta t} v_n.$$

This leads to the following typical picture of standard Brownian motion

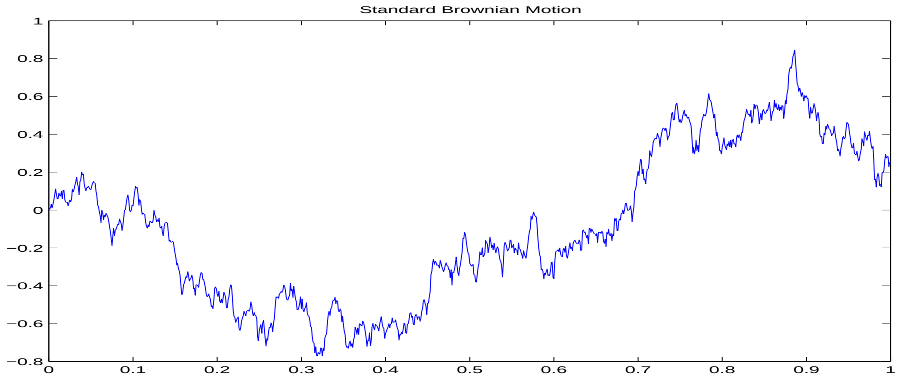

Figure 24: A sample path of a standard Brownian motion

The Matlab code for this is very simple:

```
T=1;
N=250;
dt=T/N;
tt=[0:dt:T];

bm(1)=0;
for j=2:N
    bm(j)=bm(j-1)+sqrt(dt)*normrnd(0,1);
end;
plot(tt, bm);
```

By making use of the powerful vector notation and operations one can replace the for-loop by just a single line:

```
T=1;
N=250;
dt=T/N;
tt=[0:dt:T];
bm=cumsum([0 sqrt(dt)*normrnd(0,1,1,N)]);
plot(tt, bm);
```

5.2. The Gamma Process

Note that, when X is Gamma(a, b), then for $c > 0$, X/c is Gamma(a, bc). So we need only a good generator for Gamma$(a, 1)$ random numbers. Most mathematical software programs have built in random gamma number generators. In Matlab we have the command `gamrnd`. Note Matlab uses another convention for the b-parameter: one needs to invert the b-parameter of our setting.

To simulate a sample path of a Gamma process $G = \{G_t, t \geq 0\}$, where G_t follows a Gamma(at, b) law at time points $\{n\Delta t, n = 0, 1, \dots\}$:

- generate independent Gamma$(a\Delta t, b)$ random numbers $\{g_n, n \geq 1\}$
- Then set $G_0 = 0$ and
$$G_{n\Delta t} = G_{(n-1)\Delta t} + g_n, \quad n \geq 1.$$

The Matlab code for this is very simple:

```
T=1;
N=250;
dt=T/N;
tt=[0:dt:T];
a=10; b=20;

gp(1)=0;
for j=2:N

    gp(j)=gp(j-1)+gamrnd(a*dt,1/b);

end;
plot(tt, gp);
```

By making use of the powerful vector notation and operations one can one more replace the for-loop by just a single line:

```
T=1;
N=250;
dt=T/N;
tt=[0:dt:T]
gp=cumsum([0 gamrnd(a*dt,1/b,1,N)]);
plot(tt,gp);
```

5.3. The Variance Gamma Process

5.3.1. VG as the Difference of Two Gamma Processes

A VG process is the difference of two independent Gamma processes. More precisely a VG process $X^{(VG)}$ with parameters $C, G, M > 0$ can be decomposed as

$$X_t^{(VG)} = G_t^{(1)} - G_t^{(2)},$$

where

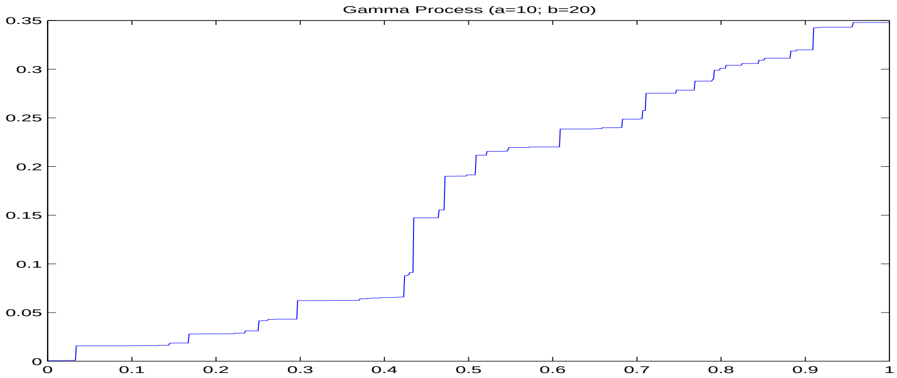

Figure 25: A sample path of a Gamma process ($a = 10$ and $b = 20$.)

- $G^{(1)}$ is a Gamma process with parameters $a = C$ and $b = M$;
- $G^{(2)}$ is a Gamma process with parameters $a = C$ and $b = G$.

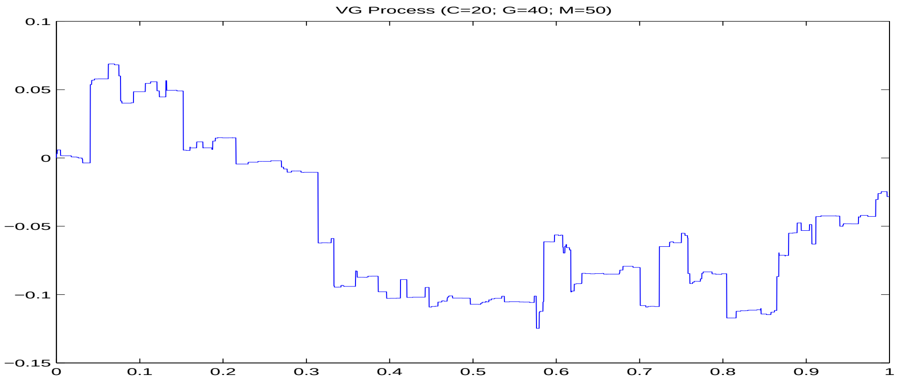

Figure 26: A sample path of a VG process $C = 20$, $G = 40$ and $M = 50$.

Matlab code for VG process as difference of two Gamma processes and the VG stock process:

```
Snul=100; T=1; r=0.04; q=0.03; n=250;
nu=0.2; sigma=0.15; theta= -0.10
omega=log(1-sigma^2*nu/2-theta*nu)/nu;
C=1/nu;
G=(sqrt(theta^2*nu^2/4+sigma^2*nu/2)-theta*nu/2)^(-1);
M=(sqrt(theta^2*nu^2/4+sigma^2*nu/2)+theta*nu/2)^(-1);

dt=T/n;
tt=[0:dt:T];

S(1)=Snul;
vg(1)=0;
for s = 1:n
```

```
    g1= gamrnd(dt*C,1/M);

    g2= gamrnd(dt*C,1/G);

    vg(s+1) = vg(s) + g1-g2;

    S(s+1) = Snul*exp((r-q+omega)*tt(s+1)+vg(s+1));

end;
```

Making use of vector notation, one could replace the last `for`-loop by

```
g1=[0,cumsum(gamrnd(dt*C,1/M,1,n))];
g2=[0,cumsum(gamrnd(dt*C,1/G,1,n))];
S = Snul*exp((r-q+omega)*tt+g1-g2);
```

5.3.2. VG as Time-Changed Brownian Motion

A VG process $X^{(VG)}$ with parameters (σ, ν, θ) out of a standard Brownian motion W with drift by a Gamma process G with parameters $a = 1/\nu$ and $b = 1/\nu$. We have

$$X_t^{(VG)} = \theta G_t + \sigma W_{G_t}.$$

VG(σ, ν, θ)-random numbers h_n can be obtained out of Gamma(ν^{-1}, ν^{-1}) random numbers g_n and Normal$(0,1)$ numbers v_n:

$$h_n = \theta g_n + \sigma \sqrt{g_n} v_n.$$

Matlab code for VG process as Gamma-time-changed Brownian Motion with drift:

```
Snul=100; T=1; r=0.04; q=0.03; n=250;
nu=0.2; sigma=0.15; theta= -0.10
omega=log(1-sigma^2*nu/2-theta*nu)/nu;

dt=T/n;
tt=[0:dt:T];

vg(1)=0;
for s = 1:n

    g= gamrnd(dt/nu,nu);

    vg(s+1) = vg(s) + theta*g+ sigma*sqrt(g)*normrnd(0,1);

end;
S = Snul*exp((r-q+omega).*tt+vg);
```

6. APPENDIX: CHARACTERISTIC FUNCTIONS

In this appendix we given more details about characteristic functions, essential mathematical objects in the financial engineering.

Characteristic functions provide us a way to describe the dynamics/stochastics of some popular advanced models unambiguously. Moreover, out of them one can obtain many interesting properties of the underlying distribution like moments. Finally and maybe for us most importantly, they serve as main input in our vanilla pricing algorithm which will be constructed in Section 4.

The characteristic function ϕ of a distribution, or equivalently of a random variable X, is the Fourier-Stieltjes transform of the distribution function $F(x) = P(X \leq x)$:

$$\phi_X(u) = E[\exp(\mathrm{i}uX)] = \int_{-\infty}^{+\infty} \exp(\mathrm{i}ux)\mathrm{d}F(x),$$

where i is the imaginary number ($\mathrm{i}^2 = -1$).

In almost all cases we will work with a random variable X that has a continuous distribution which has a density function, say $f_X(x)$. One then can write:

$$\phi_X(u) = E[\exp(\mathrm{i}uX)] = \int_{-\infty}^{+\infty} \exp(\mathrm{i}ux) f_X(x)\mathrm{d}x.$$

Example: The Normal Distribution Normal(μ, σ^2), with mean μ and variance σ^2 lives on $(-\infty, +\infty)$ and has a density function given by:

$$f(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

Its characteristic function is given by

$$\begin{aligned} \phi_X(u) &= \int_{-\infty}^{+\infty} \exp(\mathrm{i}ux) \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \mathrm{d}x \\ &= \exp(\mathrm{i}u\mu) \exp\left(-\frac{1}{2}\sigma^2 u^2\right). \end{aligned}$$

Let us prove the above formula for the Normal$(0, 1)$ case. We will comment later on how to derive the general situation. We have

$$\begin{aligned}
\phi_X(u) &= \int_{-\infty}^{+\infty} \exp(\mathrm{i}ux)\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left(-\frac{x^2}{2} + \mathrm{i}ux\right) \mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left(-\left((x/\sqrt{2})^2 - \mathrm{i}ux + (\mathrm{i}u/\sqrt{2})^2 - (\mathrm{i}u/\sqrt{2})^2\right)\right) \mathrm{d}x \\
&= \frac{\exp(-u^2/2)}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left(-\left((x/\sqrt{2} - \mathrm{i}u/\sqrt{2})^2\right)\right) \mathrm{d}x \\
&= \frac{\exp(-u^2/2)}{\sqrt{2\pi}} \sqrt{2} \int_{-\infty}^{+\infty} \exp(-z^2)\mathrm{d}z \\
&= \frac{\exp(-u^2/2)}{\sqrt{2\pi}} \sqrt{2}\sqrt{\pi} \\
&= \exp(-u^2/2),
\end{aligned}$$

where we used in the penultimate line the formula for the so-called Gauss-integral: $\int_{-\infty}^{+\infty} \exp(-z^2)\mathrm{d}z = \sqrt{\pi}$.

Example: The Gamma Distribution $Gamma(a, b)$ with parameters a and b is a distribution on the positive real line $(0, +\infty)$ and is defined by its density function:

$$f(x; a, b) = \frac{b^a}{\Gamma(a)} x^{a-1} \exp(-bx), \qquad x > 0.$$

A motivated reader can check that here the characteristic function is given by

$$\begin{aligned}
\phi_X(u) &= \int_0^{+\infty} \exp(\mathrm{i}ux) \frac{b^a}{\Gamma(a)} x^{a-1} \exp(-bx)\, \mathrm{d}x \\
&= (1 - \mathrm{i}ub^{-1})^{-a}.
\end{aligned}$$

For any distribution, the characteristic function always exists, is continuous and it determines the distribution function uniquely. Moreover we have

$$\phi(0) = E[\exp(\mathrm{i}0X)] = E[1] = 1.$$

Furthermore, we have

$$\phi_{-X}(u) = E[\exp(\mathrm{i}u(-X))] = E[\exp(\mathrm{i}(-u)X)] = \phi_X(-u). \tag{10}$$

More generally, we have that for any real constant a,

$$\phi_{aX}(u) = E[\exp(\mathrm{i}u(aX))] = E[\exp(\mathrm{i}(au)X)] = \phi_X(au).$$

From ϕ one can easily derive the moments of X. Indeed, if $E[|X|^k] < \infty$, then

$$E[X^k] = \mathrm{i}^{-k} \frac{\mathrm{d}}{\mathrm{d}u^k} \phi(u)\bigg|_{u=0}.$$

A very convenient property, is the fact that the sum of two independent random variable, or equivalently the convolution of two distributions, translate into the product of the corresponding characteristic functions. More precisely, if X and Y are two independent random variables with characteristic functions ϕ_X and ϕ_Y resp., then the characteristic function of $Z = X + Y$ is given by

$$\phi_Z(u) = \phi_X(u)\phi_Y(u). \tag{11}$$

Next, note that the degenerate random variable $X \equiv \mu$, i.e. the random variable that takes with probability one the constant value μ (so there is no randomness), has characteristic function

$$E[\exp(\mathrm{i}uX)] = \exp(\mathrm{i}u\mu).$$

Example: Using the above properties, we can easily deduce the characteristic function of a random variable Z with the Normal(μ, σ^2) distribution. Indeed, write $Z = \mu + \sigma X$ with X standard Normal. Then

$$\begin{aligned}
\phi_Z(u) &= \phi_{\mu+\sigma X}(u) \\
&= \phi_\mu(u)\phi_{\sigma X}(u) \\
&= \phi_\mu(u)\phi_X(\sigma u) \\
&= \exp(\mathrm{i}u\mu)\exp(-(\sigma u)^2/2)
\end{aligned}$$

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CONTRIBUTED TALKS

DEATH BONDS WITH STOCHASTIC FORCE OF MORTALITY

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Abstract

In a financial market with stochastic interest rate following a square root process, we present a closed form solution for pricing a death bond (as a security backed by insurance contracts) when the force of mortality follows a square root stochastic process whose expected value coincides, at any time, with the force of mortality given by the so-called Gompertz Makeham density.

1. INTRODUCTION AND CONCLUSION

The managers of either an insurance company or a pension fund are concerned about both financial and actuarial risks. Nevertheless, these two kinds of risk can be partially or fully hedged with very different instruments. For instance, the inflation risk, the interest rate risk, and the exchange rate risk, can all be efficiently hedged by existing financial assets (respectively, inflation indexed bonds, floating coupon bonds, and forwards, futures, or options on the foreign exchange). For what concerns the actuarial risk, its hedging and diversification are more difficult because of the lack of traded assets which could be able to provide their holder with cash flows correlated with the above mentioned risks. Furthermore, since the financial assets are usually very poorly (when not at all) correlated with the actuarial risk sources, then there is no portfolio of financial asset that can provide a suitable hedging against the actuarial risk. Accordingly, in order to be able to effectively hedge against actuarial risk, the institutional investors should trade, on the financial market, new assets correlated with death (or survival) probability of economic agents. Such assets wouldn't of course provide any hedging against the so-called basis risk, i.e. the risk that the population an actuarial-financial asset is written on diverges from the population whose demographic behaviour we are trying to hedge against. Nevertheless, these actuarial-financial assets would make many institutional investors able to bear the so-called longevity and mortality risks. The longevity risk could be almost perfectly hedged through longevity bonds (see, for instance, Azzoppardi 2005, Menoncin 2006, 2008) and, in the same spirit, there are nowadays rumors about the issue of some new assets called death bonds which should belong to the family of the Asset Backed Securities

(ABS). In particular, death bonds should be backed by death insurance sold by their holder in exchange of the net present value of the final benefit. One of the main concern about both longevity and mortality risk is that the force of mortality (given by the amount of people who die in a given period as a percentage of the whole population) is stochastic itself. In fact, once it has been estimated and foreseen on the basis of the actuarial tables, it is nevertheless affected by unforeseeable factors. In particular, the length of human life (for both men and women) has been significantly increasing during the last decades. Such an increase implies a serious risk for pension funds which will have to pay pensions for periods longer than that they had foreseen when entering the pension agreements with their sponsors. The force of mortality (or the survival probability) can be profitably modelling by using well known results about stochastic processes (see, for instance, Dahl 2004, Biffis 2005, Hainaut and Devolder 2008). In this paper we present a model for the stochastic force of mortality following a generalized Cox et al. (1985) process and consistent with the so-called Gompertz Makeham density function (see, for instance, Milevsky 2006). In this framework, we are able to price a death insurance/death bond in a closed form and we show that the return on a death bond is decreasing through time. The rest of the paper is structured as follows. Section 2 shows the model for the instantaneously riskless interest rate and a zero-coupon bond written as a derivative on the interest rate. Section 3 presents the model we take into account for the stochastic force of mortality. Section 4 and 5 contain the main result about pricing a death insurance and a death bond both in a stochastic framework. Section 6 concludes. The technicalities about the main results are left to appendices.

2. INTEREST RATE AND BONDS

The instantaneously riskless interest rate $r(t)$ is assumed to follow the stochastic differential equation (Cox et al. 1985)

$$\begin{aligned} dr(t) &= a_r\left(\gamma_r - r(t)\right)dt + \sigma_r\sqrt{r(t)}\,dW_r(t), \\ r\left(t_0\right) &= r_0, \end{aligned} \tag{1}$$

with positive constant r_0 and where $W_r(t)$ is a Brownian motion. For pricing purposes, we need to compute the stochastic process (1) under a risk-neutral probability measure ($\mathbb{Q}$). After Girsanov's theorem we know that on an arbitrage free financial market there exists (at least) a market price of risk ξ_r such that

$$dW_r^{\mathbb{Q}} = \xi_r dt + dW_r.$$

Here, we assume that the market price of risk is given by $\xi_r = \sqrt{r(t)}\frac{\psi}{\sigma_r}$, with ψ constant so that the process of $r(t)$ under $\mathbb{Q}$ can be written in the same form as (1):

$$dr(t) = a_r^{\mathbb{Q}}\left(\gamma_r^{\mathbb{Q}} - r(t)\right)dt + \sigma_r\sqrt{r(t)}\,dW_r^{\mathbb{Q}}, \tag{2}$$

with

$$a_r^{\mathbb{Q}} \equiv a_r + \psi, \qquad \gamma_r^{\mathbb{Q}} \equiv \frac{a_r}{a_r + \psi}\gamma_r.$$

In this framework, we can state what follows.

Proposition 2.1 *If the riskless interest rate follows (1), then the price of a zero-coupon $B(t,T)$ is given by*

$$B(t,T) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r(s)ds}\right] = e^{-a_r^{\mathbb{Q}}\gamma_r^{\mathbb{Q}}\int_t^T C_B(s,T)ds - C_B(t,T)r(t)}, \tag{3}$$

where

$$\begin{aligned} C_B(t,T) &= 2\frac{1-e^{-k(T-t)}}{k+a_r^{\mathbb{Q}}+\left(k-a_r^{\mathbb{Q}}\right)e^{-k(T-t)}}, \\ k &\equiv \sqrt{\left(a_r^{\mathbb{Q}}\right)^2+2\sigma_r^2} \end{aligned} \tag{4}$$

whose differential is

$$\begin{aligned} \frac{dB(t,T)}{B(t,T)} &= r(t)dt - C_B(t,T)\sigma_r\sqrt{r(t)}\,dW_r^{\mathbb{Q}}(t) \\ &= \left(r(t) - C_B(t,T)\sigma_r\sqrt{r(t)}\xi_r\right)dt - C_B(t,T)\sigma_r\sqrt{r(t)}\,dW_r(t). \end{aligned} \tag{5}$$

Proof. See Appendix A with $X = r$ and $B(t,T) = V(t,T)|_{\chi=0}$. ∎

Another way to write the value of a zero-coupon is to use the forward (instantaneous) interest rate $f(t,T)$:

$$B(t,T) = e^{-\int_t^T f(t,s)ds},$$

where we do not need any longer the expected value since the whole curve of the forward rate $f(t,s)$ is known in t for any $s \geq t$.

An obvious no-arbitrage condition asks for the expected discounted value of $r(T)$ to equate the expected discounted value of $f(t,T)$:

$$\mathbb{E}_t^{\mathbb{Q}}\left[r(T)e^{-\int_t^T r(s)ds}\right] = \mathbb{E}_t^{\mathbb{Q}}\left[f(t,T)e^{-\int_t^T r(s)ds}\right],$$

but since $f(t,T)$ belongs to the information set (i.e. $\sigma-$algebra) in t, then we have

$$f(t,T) = \frac{\mathbb{E}_t^{\mathbb{Q}}\left[r(T)e^{-\int_t^T r(s)ds}\right]}{\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r(s)ds}\right]}.$$

Proposition 2.2 *If the riskless interest rate $r(t)$ follows the process (1), then the forward interest rate is given by*

$$f(t,T) = \int_t^T a_r^{\mathbb{Q}}\gamma_r^{\mathbb{Q}} e^{-\int_s^T\left(a_r^{\mathbb{Q}}+C_B(u,T)\sigma_r^2\right)du}ds + e^{-\int_t^T\left(a_r^{\mathbb{Q}}+C_B(u,T)\sigma_r^2\right)du}r(t), \tag{6}$$

where $C_B(t,T)$ is as in (4), and whose differential is

$$\begin{aligned} df(t,T) &= e^{-\int_t^T\left(a_r^{\mathbb{Q}}+C_B(u,T)\sigma_r^2\right)du}C_B(t,T)\sigma_r^2 r(t)dt \\ &\quad + e^{-\int_t^T\left(a_r^{\mathbb{Q}}+C_B(u,T)\sigma_r^2\right)du}\sigma_r\sqrt{r(t)}\,dW_r^{\mathbb{Q}}, \end{aligned}$$

where it is true that

$$\frac{\partial C_B(t,T)}{\partial T} = e^{-\int_t^T\left(a_r^{\mathbb{Q}}+C_B(u,T)\sigma_r^2\right)du}.$$

Proof. See Appendix A with $X = r$ and $f(t,T) = \frac{V(t,T)|_{\chi=1}}{V(t,T)|_{\chi=0}}$. ∎

We will show these results have a straight parallel in an actuarial framework.

3. MORTALITY RATE

Let us call τ the (stochastic) death time whose density function is $\pi(\tau)$. In this way the probability of surviving from t_0 up to t, for an agent aged t_0, is given by

$$\left({}_tp_{t_0}\right) = 1 - \int_{t_0}^{t} \pi(s)ds,$$

whose differential is

$$\frac{d\left({}_tp_{t_0}\right)}{\left({}_tp_{t_0}\right)} = -\frac{\pi(t)}{1 - \int_{t_0}^{t} \pi(s)ds} dt \equiv -\lambda(t)dt, \tag{7}$$

with the natural boundary condition $\left({}_{t_0}p_{t_0}\right) = 1$. In the actuarial literature, $\lambda(t)$ is often called *mortality rate* (or *hazard rate*). The (unique) solution of the ordinary differential equation (7) is

$$\left({}_tp_{t_0}\right) = e^{-\int_{t_0}^{t} \lambda(s)ds}. \tag{8}$$

One of the most common parametrizations for the mortality rate is the so called Gompertz-Makeham function:

$$\lambda(t) = \phi + \frac{1}{b} e^{\frac{t-m}{b}}, \tag{9}$$

where the positive constant have the following meaning: (i) ϕ captures the age independent component of mortality rate (like accidents), (ii) m measures the modal value of life, and (iii) b is the dispersion parameter of life. Typical value for these parameters (consistently chosen with Milevsky, 2006) are

$$\phi = 0.001, \quad m = 82.3, \quad b = 11.4. \tag{10}$$

In the real world the mortality rate does not behave in a deterministic way. In fact, it may change for unforeseeable reasons. In order to fix that, we can model $\lambda(t)$ as a stochastic process itself:

$$\begin{aligned} d\lambda(t) &= \mu_\lambda(t, \lambda)dt + \sigma_\lambda(t, \lambda)dW(t), \\ \lambda(t_0) &= \lambda_0. \end{aligned}$$

In this case the survival probability cannot be written as in (8). In fact, under the information set in t, we do not know all the mortality rates from t to T. Accordingly, we can compute the survival probability from t_0 to t only under the expected value conditional to the information set in t_0:

$$\left({}_tp_{t_0}\right) = \mathbb{E}_{t_0}\left[e^{-\int_{t_0}^{t} \lambda(s)ds}\right]. \tag{11}$$

We highlight that, in this case, the expected value is computed under the historical probability measure (we haven't put any upper script on the expected value), and not under the risk neutral probability.

If we are allowed to differentiate with respect to time (t) under the expected value[1], then we can conclude from (7) that

$$\pi(t) = -\frac{d\left({}_tp_{t_0}\right)}{dt} = \mathbb{E}_{t_0}\left[\lambda(t)e^{-\int_{t_0}^{t} \lambda(s)ds}\right].$$

[1]Grandell (1976) shows that the equality which follows is true if: (i) there exists a constant C such that, for any t, $\mathbb{E}_{t_0}\left[\lambda(t)^2\right] < C$, and (ii) for any $\varepsilon > 0$ and almost every time t, $\lim_{\delta \to 0} \mathbb{P}\left(\left|\lambda\left(t+\delta\right) - \lambda(t)\right| \geq \varepsilon\right) = 0$.

In this case the hazard rate is given by

$$l(t_0,t) \equiv -\frac{d\left({}_tp_{t_0}\right)}{dt}\frac{1}{\left({}_tp_{t_0}\right)} = \frac{\mathbb{E}_{t_0}\left[\lambda(t)e^{-\int_{t_0}^{t}\lambda(s)ds}\right]}{\mathbb{E}_{t_0}\left[e^{-\int_{t_0}^{t}\lambda(s)ds}\right]}, \tag{12}$$

which coincides with $\lambda(t)$ if and only if $\lambda(t)$ is not stochastic (of course it is true, in any case, that $l\left(t_0,t_0\right) = \lambda(t_0)$).

Such a framework has a straightforward and appealing parallel with the financial framework we have already presented above. The main difference is that the expected value on the financial market is computed under the riskless probability measure ($\mathbb{Q}$) which is different from the probability measure used for computing $\left({}_tp_{t_0}\right)$ (see table 1).

Actuarial market	**Financial market**
Death intensity $\lambda(t)$	Interest rate $r(t)$
Survival probability $\left({}_tp_{t_0}\right)$	Zero-coupon bond price $B(t_0,t)$
Hazard rate $l(t_0,t)$	Instant forward rate $f(t_0,t)$
Computations under $\mathbb{P}$	Computations under $\mathbb{Q}$

Table 1: Correspondences between actuarial and financial frameworks

Now, we want to build a stochastic process for the variable $\lambda(t)$ such that its expected value is, at any instant, equal to the Gompertz-Makeham mortality rate (9). For this purpose, we use the following result.

Proposition 3.1 *If the stochastic variable $X(t)$ solves*

$$\begin{aligned} dX(t) &= \alpha(t)\left(\frac{1}{\alpha(t)}\frac{\partial\beta(t)}{\partial t} + \beta(t) - X(t)\right)dt + \sigma(t,X)dW(t), \\ X(t_0) &= \beta(t_0), \end{aligned}$$

then

$$\mathbb{E}_{t_0}\left[X(t)\right] = \beta(t).$$

Proof. Let us apply Itô's lemma to $Y(t) = X(t)e^{\int_{t_0}^{t}\alpha(u)du}$:

$$\begin{aligned} dY(t) &= e^{\int_{t_0}^{t}\alpha(u)du}dX(t) + \alpha(t)X(t)e^{\int_{t_0}^{t}\alpha(u)du}dt \\ &= \frac{\partial}{\partial t}\left(\beta(t)e^{\int_{t_0}^{t}\alpha(u)du}\right)dt + e^{\int_{t_0}^{t}\alpha(u)du}\sigma(t,X)dW(t). \end{aligned}$$

Now, we compute the expected value under the information set in t_0:

$$\mathbb{E}_{t_0}\left[dY(t)\right] = \frac{\partial}{\partial t}\left(\beta(t)e^{\int_{t_0}^{t}\alpha(u)du}\right)dt,$$

and by integrating from t_0 up to t we have

$$\mathbb{E}_{t_0}\left[Y(t)\right] = Y(t_0) + \beta(t)e^{\int_{t_0}^{t}\alpha(u)du} - \beta(t_0).$$

After substituting for Y we finally obtain

$$\mathbb{E}_{t_0}\left[X(t)\right] = \left(X(t_0) - \beta(t_0)\right) e^{-\int_{t_0}^{t} \alpha(u)du} + \beta(t),$$

and, since $X(t_0) = \beta(t_0)$, the result of the proposition is obtained. ■

We want the expected value of $\lambda(t)$ to be always equal to the Gompertz function (9), i.e.

$$\mathbb{E}_{t_0}\left[\lambda(t)\right] = \phi + \frac{1}{b} e^{\frac{t-m}{b}}. \tag{13}$$

By using the result of Proposition 3.1, we can write the process for $\lambda(t)$ as

$$\begin{aligned} d\lambda(t) &= a_\lambda \left(\gamma_\lambda(t) - \lambda(t)\right) dt + \sigma_\lambda \sqrt{\lambda(t)}\, dW_\lambda(t), \qquad (14) \\ \lambda(t_0) &= \phi + \frac{1}{b} e^{\frac{t_0-m}{b}}, \end{aligned}$$

where

$$\gamma_\lambda(t) \equiv \phi + \left(\frac{1}{a_\lambda b} + 1\right) \frac{1}{b} e^{\frac{t-m}{b}}, \tag{15}$$

and a_λ and σ_λ are two constant (and positive) parameters that can be estimated from the historical series on $\lambda(t)$.

In order to trace our model back to the well known results about the affine stochastic processes, we have chosen to set the diffusion term as the square root of the stochastic variable $\lambda(t)$ itself.

Since we need the force of mortality to be always positive, we give now a condition under which $\lambda(t)$ never becomes negative.

Proposition 3.2 *If*

$$\sigma_\lambda^2 \leq 2a_\lambda \left(\phi + \left(\frac{1}{a_\lambda b} + 1\right) \frac{1}{b} e^{\frac{t_0-m}{b}}\right),$$

then the value of $\lambda(t)$ in (14) never becomes negative.

Proof. See Appendix B. ■

Proposition 3.3 *If the death intensity $\lambda(t)$ follows the process (14), then the survival probability is given by*

$$\left({}_T p_t\right) = e^{-a_\lambda \int_t^T C_P(s,T)\gamma_\lambda(s)ds - C_P(t,T)\lambda(t)}, \tag{16}$$

where

$$\begin{aligned} C_P(t,T) &= 2\frac{1 - e^{-k(T-t)}}{k + a_\lambda + \left(k - a_\lambda\right) e^{-k(T-t)}}, \qquad (17) \\ k &\equiv \sqrt{a_\lambda^2 + 2\sigma_\lambda^2}, \end{aligned}$$

and, in differential terms,

$$\frac{d\left({}_T p_t\right)}{\left({}_T p_t\right)} = \lambda(t)dt - C_P(t,T)\sigma_\lambda \sqrt{\lambda(t)}\, dW_\lambda(t).$$

Proof. See Appendix A. ■

Proposition 3.4 *If the death intensity $\lambda(t)$ follows the process (14), then the hazard rate is given by*

$$l(t,T) = \int_t^T a_\lambda \gamma_\lambda(s) e^{-\int_s^T \left(a_\lambda + C_P(u,T)\sigma_\lambda^2\right)du} ds + e^{-\int_t^T \left(a_\lambda + C_P(u,T)\sigma_\lambda^2\right)du} \lambda(t), \quad (18)$$

whose differential is

$$\begin{aligned} dl(t,T) &= e^{-\int_t^T \left(a_\lambda + C_P(u,T)\sigma_\lambda^2\right)du} C_P(t,T)\sigma_\lambda^2 \lambda(t)dt \\ &\quad + e^{-\int_t^T \left(a_\lambda + C_P(u,T)\sigma_\lambda^2\right)du} \sigma_\lambda \sqrt{\lambda(t)}\, dW_\lambda(t), \end{aligned} \quad (19)$$

where the function $C_P(t,T)$ is as in (17).

Proof. See Appendix A. ■

4. DEATH INSURANCE

The subscriber of a death insurance agrees to pay settlements (P) during his lifetime in order to receive, at his death, a given amount of money (final benefit). For the sake of simplicity we will set such an amount to 1

If the death insurance is subscribed in t_0 (i.e. when the subscriber is aged t_0), then the actuarial equilibrium for such a contract asks for the expected present value of the settlements to equate the expected present value of the final benefit (available at the death time τ and equal to 1). If we assume that P is continuously paid, then the actuarial equilibrium asks for the following equality to hold:

$$\mathbb{E}_{t_0}^{\tau}\left[\int_{t_0}^{\tau} P(s)e^{-\int_{t_0}^{s} r_\lambda(u)du} ds\right] = \mathbb{E}_{t_0}^{\tau}\left[e^{-\int_{t_0}^{\tau} r_\lambda(u)du}\right],$$

where r_λ is a suitable discount rate used by the insurance company.

When the insurance contract enters (in any way) the financial market, then the value of the contract must be computed as the value of any other asset i.e. under the risk neutral probability measure. Accordingly, the discount rate r_λ is replaced by the riskless interest rate r as follows

$$\mathbb{E}_{t_0}^{\mathbb{Q},\tau}\left[\int_{t_0}^{\tau} P(s)e^{-\int_{t_0}^{s} r(u)du} ds\right] = \mathbb{E}_{t_0}^{\mathbb{Q},\tau}\left[e^{-\int_{t_0}^{\tau} r(u)du}\right].$$

Now, as it is usually the case, the riskless interest rate r is assumed to be independent of the death time τ. Accordingly, the expected value computed under $\mathbb{Q}$ and τ can be separately computed, and the equilibrium condition becomes

$$\int_{t_0}^{\infty} \mathbb{E}_{t_0}^{\mathbb{Q},\tau}\left[\mathbb{I}_{s<\tau}\right] \mathbb{E}_{t_0}^{\mathbb{Q}}\left[P(s)e^{-\int_{t_0}^{s} r(u)du}\right] ds = \int_{t_0}^{\infty} \mathbb{E}_{t_0}^{\mathbb{Q},\tau}\left[\pi(s)\right] \mathbb{E}_{t_0}^{\mathbb{Q}}\left[e^{-\int_{t_0}^{s} r(u)du}\right] ds$$

where $\mathbb{I}_\varepsilon$ is the indicator function whose value is 1 if the event ε happens and 0 otherwise. Since the expected value of the indicator function coincides with the probability of ε then we have

$$\begin{aligned}&\int_{t_0}^{\infty}\mathbb{E}_{t_0}^{\mathbb{Q}}\left[e^{-\int_{t_0}^{s}\lambda(u)du}\right]\mathbb{E}_{t_0}^{\mathbb{Q}}\left[P(s)e^{-\int_{t_0}^{s}r(u)du}\right]ds\\ =&\int_{t_0}^{\infty}\mathbb{E}_{t_0}^{\mathbb{Q}}\left[\lambda(s)e^{-\int_{t_0}^{s}\lambda(u)du}\right]\mathbb{E}_{t_0}^{\mathbb{Q}}\left[e^{-\int_{t_0}^{s}r(u)du}\right]ds,\end{aligned}$$

and, if P is constant we have

$$P^* = \frac{\int_{t_0}^{\infty}\mathbb{E}_{t_0}^{\mathbb{Q}}\left[\lambda(s)e^{-\int_{t_0}^{s}\lambda(u)du}\right]B(t_0,s)ds}{\int_{t_0}^{\infty}\mathbb{E}_{t_0}^{\mathbb{Q}}\left[e^{-\int_{t_0}^{s}\lambda(u)du}\right]B(t_0,s)ds}.$$

By recalling (12) and (11),we can finally write

$$P^* = \frac{\int_{t_0}^{\infty} l(t_0,t)^{\mathbb{Q}}\left({}_tp_{t_0}\right)^{\mathbb{Q}}B(t_0,s)ds}{\int_{t_0}^{\infty}\left({}_tp_{t_0}\right)^{\mathbb{Q}}B(t_0,s)ds}, \tag{20}$$

where we have indicated with $\left({}_tp_{t_0}\right)^{\mathbb{Q}}$ and $l\left(t_0,t\right)^{\mathbb{Q}}$ the survival probability and the hazard rate respectively, computed under the risk neutral probability.

Once the value of the premium is obtained, the value of the death insurance, at any time t, is given by the difference between the expected present value of the final benefit and the expected present value of the premia still due:

$$D(t) = \mathbb{E}_t^{\mathbb{Q},\tau}\left[e^{-\int_t^{\tau}r(u)du}\right] - P^*\mathbb{E}_t^{\mathbb{Q},\tau}\left[\int_t^{\tau}e^{-\int_t^{s}r(u)du}ds\right],$$

which can be simplified as we have done above by obtaining

$$D(t) = \int_t^{\infty}\left(l(t,s)^{\mathbb{Q}} - P^*\right)\left({}_sp_t\right)^{\mathbb{Q}}B(t,s)ds. \tag{21}$$

As it is evident from this last equation, the value of the premium for a death insurance subscribed in t_0 can also be computed from (21) by imposing the condition $D(t_0) = 0$.

From Equation (21) it is evident that the death insurance cannot be distinguished from an infinitely living bond whose coupons are given by the difference between the hazard rate and the equilibrium premium, weighted by the survival probability.

The differential form of (21) is

$$\begin{aligned}\frac{dD(t)}{D(t)} =& \left(r(t)+\lambda(t)+\frac{P^*-\lambda(t)}{D(t)}\right)dt\\ &+\frac{D_r(t)}{D(t)}\sigma_r\sqrt{r}\,dW_r^{\mathbb{Q}}+\frac{D_\lambda(t)}{D(t)}\sigma_\lambda\sqrt{\lambda}\,dW_\lambda^{\mathbb{Q}},\end{aligned} \tag{22}$$

where D_λ and D_r are the partial derivatives of D with respect to λ and r respectively and, in particular,

$$\begin{aligned}\frac{\partial D(t)}{\partial\lambda(t)} =& \int_t^{\infty}e^{-\int_t^{s}\left(a_\lambda+C_P(u,s)\sigma_\lambda^2\right)du}\left({}_sp_t\right)^{\mathbb{Q}}B(t,s)ds\\ &-\int_t^{\infty}C_P(t,s)\left(l(t,s)^{\mathbb{Q}}-P^*\right)\left({}_sp_t\right)^{\mathbb{Q}}B(t,s)ds,\end{aligned}$$

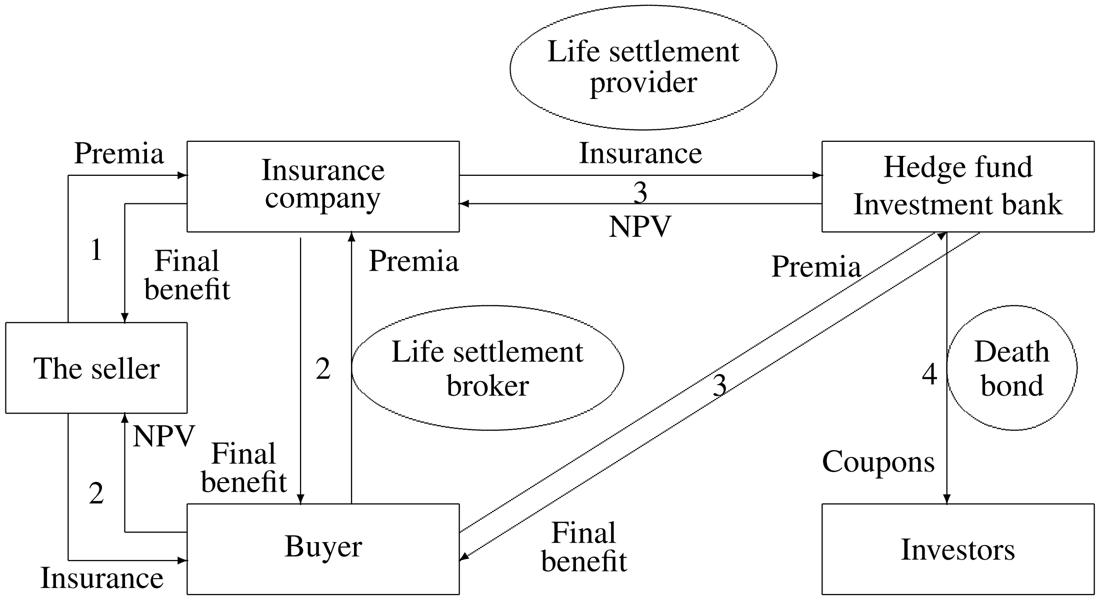

Figure 1: How a death insurance becomes a death bond

$$\frac{\partial D(t)}{\partial r(t)} = -\int_t^{\infty} C_B\,(t,s)\left(l\,(t,s)^{\mathbb{Q}} - P^*\right)({}_s p_t)^{\mathbb{Q}}\,B(t,s)ds.$$

5. DEATH BOND

A death bond belongs to the family of the *Asset Backed Security* (ABS). The process for changing a death insurance into a death bond is made by 4 steps (as in figure 1). Let us see such steps in details.

1. The so-called seller is the subscriber of the death insurance. When the agent becomes older (typically 70) and he does not have any further need for the insurance on his life, he would like to cash out his policy.

2. The seller hires a *life settlement* broker who will find a buyer for his policy who pays the net present value of the policy (we have called $D(t)$ in the previous section). Thus, the buyer will continue paying the settlements to the insurance company and he will also receive the final benefit when the seller dies. The up-front payout to the seller varies widely, from 20% of the death benefit to 40%. The seller pays to the broker a commission ranged from 5% to 6%.

3. Another character in this game is the so-called *life settlement provider*. Through him, a hedge fund or an investment bank buys a pool of death insurances from insurance companies. Now, the hedge fund will receive the premia from the buyer and will pay the final benefit.

4. In the last step, after a sufficient number of policies has been collected, these policies can back the emission of a death bond. Accordingly, the policies play the same role as the

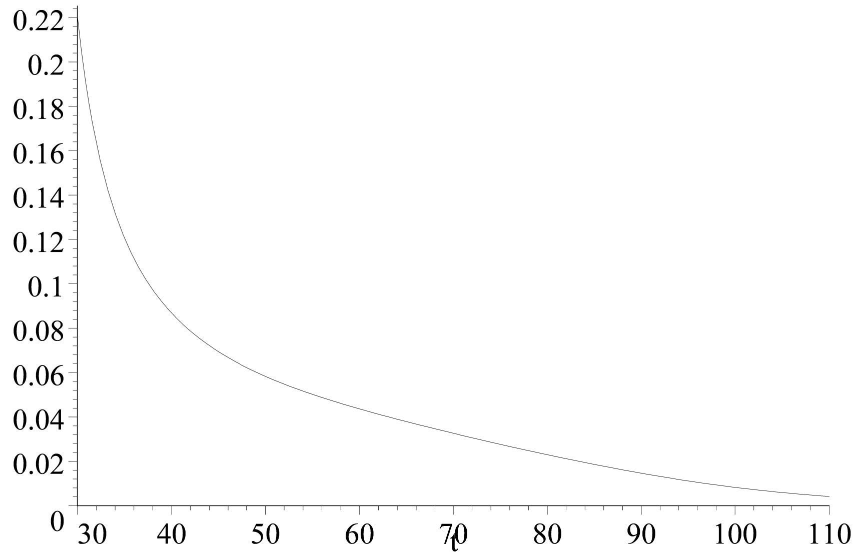

Figure 2: Return on a death bond ($\frac{dD}{D}$) with $t_0 = 25$ and $r = 0.05$ for t going from 30 to 110.

assets in an ABS or the mortgages in a mortgage backed security. The new death bond is a *pass through* asset: the premia received by the hedge fund are directly paid to the investors (without any guarantee provided by the hedge fund).

From Equation (22) we can see that the return on a death bond is given by $r(t) + \lambda(t) + \frac{P^* - \lambda(t)}{D(t)}$. In a fully deterministic case with the values of the parameters given in (10), $r = 0.05$, and $t_0 = 25$, the premium is given by $P^* = 0.0066$ and the return on the death bond for time t going from 30 to 110 is plotted in figure 2.

It is evident that the bond return decreases while time goes on. In fact, the best case for the buyer of the death bond is when the seller immediately dies after receiving the first premium P^*.

A. COMPUTATION OF $\mathbb{E}_t \left[(1 - \chi + \chi X(t)) \, e^{-\int_t^T X(s) ds} \right]$

If the stochastic variable $X(t)$ follows the process

$$\begin{aligned} dX(t) &= a \left(\gamma(t) \right) - X(t)) \, dt + \sigma \sqrt{X(t)} \, dW(t), \\ X(t_0) &= X_0, \end{aligned} \tag{23}$$

then the expected value

$$V(t,T) = \mathbb{E}_t \left[(1 - \chi + \chi X(T)) \, e^{-\int_t^T X(s) ds} \right],$$

must solve the partial differential equation

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial X} a \left(\gamma(t) - X \right) + \frac{1}{2} \frac{\partial^2 V}{\partial X^2} \sigma^2 X = XV,$$

with the boundary condition

$$V(T,T) = 1 - \chi + \chi X(T),$$

where the parameter χ can take either value 1 or value 0. If $\chi = 0$, then the function V coincides with the probability $({}_T p_t)$ if $X = \lambda$ and with the value of a zero-coupon if $X = r$. Instead, if $\chi = 1$, then the function V coincides with the numerator of $l\,(t,T)$ in Equation (12).

Now we use the guess function

$$V(t,X) = (E(t) + F(t)X)\, e^{-A(t)-C(t)X},$$

where the function A, C, E, and F must be computed in order to solve the previous differential equation. The boundary condition translates into the following conditions:

$$E(T) = 1 - \chi, \quad F(T) = \chi, \quad A(T) = 0, \quad C(T) = 0.$$

Once the partial derivatives of V are substituted into the differential equation we obtain a second order polynomial in X. Since we want it to be identically zero, then the coefficients of its terms must be zero. This means that we can split the differential equation into three differential equations as follows[2]

$$\begin{cases} 0 = \dfrac{\partial E}{\partial t} + Fa\gamma(t) - E\left(A_t + Ca\gamma(t)\right), \\ 0 = \dfrac{\partial F}{\partial t} - F\left(A_t + Ca\gamma(t)\right) - Fa - CF\sigma^2, \\ 0 = -\dfrac{\partial C}{\partial t} + aC + \frac{1}{2}C^2\sigma^2 - 1. \end{cases} \tag{24}$$

We immediately see that the value of function $C(t)$ can be computed from the third equation. With the suitable boundary condition the only solution of the differential equation for $C(t)$ is given by

$$\begin{aligned} C\,(t,T) &= 2\frac{1 - e^{-k(T-t)}}{k + a + (k-a)e^{-k(T-t)}}, \\ k &\equiv \sqrt{a^2 + 2\sigma^2}. \end{aligned}$$

The values of all the other functions can be written as functions of $C(t,T)$. Now, if we wanted to compute just the survival probability, then we would have $E = 1$ and $F = 0$ with the function A accordingly solving

$$0 = \frac{\partial A}{\partial t} + Ca\gamma(t),$$

with the boundary condition $A(T) = 0$. The only solution of this equation is

$$A(t) = a\int_t^T C(s)\gamma(s)ds.$$

[2]For the sake of simplicity, we have omitted the functional dependences (except for the function $\gamma(t)$).

Given this value for $A(t)$, the two first equations of system (24) become

$$\begin{aligned} 0 &= \frac{\partial E(t)}{\partial t} + F(t)a\gamma(t), \\ 0 &= \frac{\partial F(t)}{\partial t} - F(t)\left(a + C(t)\sigma^2\right). \end{aligned}$$

We now compute the value of F from the second equation by obtaining

$$F(t) = \chi e^{-\int_t^T \left(a+C(s)\sigma^2\right)ds},$$

and the value of E can then be computed from the first equation

$$E(t) = 1 - \chi + a\int_t^T F(s)\gamma(s)ds.$$

Finally, we can write

$$\begin{aligned} V(t,T) &= \left(1 - \chi + \chi\int_t^T a\gamma(s)e^{-\int_s^T \left(a+C(u)\sigma^2\right)du}ds + \chi e^{-\int_t^T \left(a+C(u)\sigma^2\right)du}X(t)\right) \\ &\quad \times e^{-a\int_t^T C(u)\gamma(u)du - C(t)X(t)}. \end{aligned}$$

The two values we are interested into are given by

$$V(t,T)\big|_{\chi=0} = e^{-a\int_t^T C(u)\gamma(u)ds - C(t)X(t)},$$

and

$$\frac{V(t,T)\big|_{\chi=1}}{V(t,T)\big|_{\chi=0}} = \int_t^T a\gamma(s)e^{-\int_s^T \left(a+C(u)\sigma^2\right)du}ds + e^{-\int_t^T \left(a+C(u)\sigma^2\right)du}X(t).$$

B. PROOF OF PROPOSITION 3.2

If $X(t)$ follows (23) with $\gamma(t))$ constant, then it is well known that $X(t)$ never becomes negative if $\sigma^2 \leq 2a\gamma$. In order to prove the proposition, we use the following result.

Proposition B.1 *Let us assume we have two continuous, adapted processes $X_i(t), i = 1, 2$, such that*

$$X_i(t) = X_i(t_0) + \int_{t_0}^t \mu_i\left(s, X_i(s)\right)ds + \int_{t_0}^t \sigma\left(s, X_i(s)\right)dW(s),$$

and $\forall t \in [t_0, \infty[, x \in \mathbb{R}, y \in \mathbb{R}$: (i) the coefficients $\mu_i(t,x)$ and $\sigma(t,x)$ are continuous, real-valued functions, (ii) $|\sigma(t,x) - \sigma(t,y)| \leq h(|x-y|)$ where $h : [0,\infty[\times [0,\infty[$ is a strictly increasing function with $h(0) = 0$ and $\int_{(0,\varepsilon)} h^{-2}(u)du = \infty$, $\forall\varepsilon > 0$, (iii) $X_1(t_0) \leq X_2(t_0)$ a.s., (iv) $\mu_1(t,x) \leq \mu_2(t,x)$, (v) there exists a positive constant K such that either $\mu_1(t,x)$ or $\mu_2(t,x)$ satisfies $|\mu_i(t,x) - \mu_i(t,y)| \leq K|x-y|$. Then

$$\mathbb{P}\left\{X_1(t) \leq X_2(t),\ \forall t \in [t_0, \infty[\right\} = 1.$$

Proof. See Karatzas and Shreve (1991), Proposition 2.18 p. 293. ■

Here, we take the following process

$$\begin{aligned}\lambda_1(t) &= \left(\phi + \frac{1}{b}e^{\frac{t_0-m}{b}}\right) + \int_{t_0}^{t} a\left(\gamma(t_0) - \lambda_1(s)\right) ds + \int_{t_0}^{t} \sigma\sqrt{\lambda_1(s)}\, dW(s),\\ \lambda_2(t) &= \left(\phi + \frac{1}{b}e^{\frac{t_0-m}{b}}\right) + \int_{t_0}^{t} a\left(\gamma(s) - \lambda_2(s)\right) ds + \int_{t_0}^{t} \sigma\sqrt{\lambda_2(s)}\, dW(s),\end{aligned}$$

where $\gamma(t)$ is defined in (15). It is evident that both the drift and the diffusion terms respect all the conditions in Proposition B.1.

Since we have set $\lambda_1(t_0) = \lambda_2(t_0)$ and we know that $\lambda_1(t)$ never becomes negative if

$$\sigma^2 \leq 2a\gamma(t_0), \tag{25}$$

then we also know that $\lambda_2(t)$ never becomes negative if its drift is greater than $\lambda_1(t)$'s:

$$a\left(\gamma(t) - \lambda(t)\right) \geq a\left(\gamma(t_0) - \lambda(t)\right),$$

for any real λ and for any $t \in [t_0, \infty[$. Such inequality holds if and only if $\gamma(t) \geq \gamma(t_0)$. Nevertheless, since $\gamma(t)$ is strictly increasing in t, then this inequality always holds. This means that $\lambda_2(t)$ never becomes negative if just (25) holds.

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GENERIC PRICING OF FX, INFLATION AND STOCK OPTIONS UNDER STOCHASTIC INTEREST RATES AND STOCHASTIC VOLATILITY

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Abstract

We consider the pricing of FX, inflation and stock options under stochastic interest rates and stochastic volatility, for which we use a generic multi-currency framework. We allow for a general correlation structure between the drivers of the volatility, the inflation index, the domestic (nominal) and the foreign (real) rates. Having the flexibility to correlate the underlying FX/Inflation/Stock index with both stochastic volatility and stochastic interest rates yields a realistic model, which is of practical importance for the pricing and hedging of options with long-term exposures. We derive explicit pricing formulas for various securities, such as vanilla call options, forward starting options, inflation-indexed swaps and inflation caps. We consider a calibration example to FX market data and finally we conclude.

1. INTRODUCTION

The markets for long maturity and hybrid derivatives are developing more and more. Not only are increasingly exotic structures created, also the markets for plain vanilla derivatives are growing. One of the recent advances is the development of long maturity option markets across various asset classes. During the last years, long maturity securities, such as Target Auto Redemption Notes (TARN) equity-interest rate options (e.g. see Caps (2007)), Power-Reverse Dual-Currency (PRDC) Foreign Exchange (FX) swaps (e.g. see Piterbarg (2005)) and inflation-indexed Limited Price Indices (LPI) structures (e.g see Brigo and Mercurio (2006)) have become increasingly popular. Whereas for FX, inflation and hybrid structures –which explicitly depend on future interest rates evolutions– it is immediately clear that the use of stochastic interest rates is crucial in a derivative pricing model, the addition of stochastic rates is also important for the pricing and in particularly the hedging of long maturity equity derivatives (e.g. see Bakshi et al. (2000)).

Most investment banks have now standardized a three-factor modeling framework to price cross-currency (i.e., FX and inflation) options (e.g. see Sippel and Ohkoshi (2002) or Jarrow and

Yildirim (2003)). The index then follows a log-normal process, and the interest rates of both currencies are driven by one-factor Gaussian models (e.g. see Hull and White (1993)). The choice of Gaussian assumptions for the interest rates and the log-normality for the index has allowed for a very efficient, essentially closed-form, calibration to at-the-money options on the index, i.e., on the FX-rate or stock price. The assumption of log-normality for an index, though technically very convenient, does not find a justification in the financial equity markets (e.g. see Bakshi et al. (1997)), nor in the FX markets (e.g. see Piterbarg (2005)) nor in the inflation markets (e.g. see Mercurio and Moreni (2006)). In fact, the markets for these products exhibit a strong volatility skew or smile, implying log index returns deviating from normality and suggesting the use of skewed and heavier tailed distributions. Moreover, many multi-currency structures (like LPIs or PRDCs) are particularly sensitive to volatility skews/smiles as they often incorporate multiple strikes as well as callable/knockout components.

Hence, appropriate exotic option pricing models, which need to quantify the volatility exposure in such structures, should at least be able to incorporate the smiles/skews in the vanilla markets. While various methods exist to incorporate volatility smiles (i.e., local volatility, stochastic volatility and/or jumps), the calibration of such models is by no means trivial. Normally, a skew-mechanism is applied to the forward index price (i.e., the FX-rate, CPI/Equity index), however, to price multi-currency options also a term-structure involving various time points of the forward index is required. The incorporation of stochastic interest rates makes the connection between the two particularly non-trivial (e.g. see Piterbarg (2005) or Antonov et al. (2008)). Though the issue is important, Piterbarg (2005) even dubs it as 'perhaps even the most important current outstanding problems for quantitative research departments worldwide', there is remarkably little literature available on the subject even though the problem attracted both the attention of practitioners as well as from academia (e.g. see van der Ploeg (2007)).

Only very recently, a few approaches were suggested. A local volatility approach is used in Piterbarg (2005), who derives approximating formulas for calibration. Andreasen (2006) combines Heston (1993) stochastic volatility with independent stochastic interest rates drivers and derives closed-form Fourier expressions for vanilla options. To correlate the independent rate drivers with the FX-rate, Andreasen (2006) uses an indirect approach in the form of a volatility displacement parameter, which has some disadvantages as that it can lead to extreme model parameters (e.g. see Antonov et al. (2008)). The calibration of FX options' stochastic interest rates with Heston (1993) stochastic volatility under a full correlation structure is undertaken in Antonov et al. (2008) who use Markovian projection to derive approximation formulas. Though their projection technique is elegant, the quality of their approximation deteriorates for larger maturities or more extreme model parameters. The exact pricing of FX options under Schöbel and Zhu (1999) stochastic volatility, single-factor Gaussian rates and a full correlation structure was recently considered in van Haastrecht et al. (2008).

In this paper, building on the results of van Haastrecht et al. (2008), Andreasen (2006) and Piterbarg (2005), we consider the pricing of foreign exchange, inflation and stock options under Schöbel and Zhu (1999) and Heston (1993) stochastic volatility and under multi-factor Gaussian interest rates with a full correlation structure. Since stock and FX options are special (nested) cases of inflation-indexed caps/floors[1], we will mainly focus on the pricing of inflation index derivatives.

[1]In our framework an inflation option can be seen as forward-starting FX-option, hence the pricing of FX-option follows from the pricing of inflation option by setting the forward starting date equal to the current date. A stock

The stock and FX model option pricing formulas hence follow directly from our generalization of the foreign exchange inflation framework of Jarrow and Yildirim (2003).

The setup of the paper is as follows. In Section 2, we introduce our new model. The corresponding pricing methodology is considered in Section 3.1, while in Section 3.2 we derive the characteristic functions required for the Fourier-based pricing methods under Schöbel and Zhu (1999) stochastic volatility, and touch upon the case with Heston (1993) stochastic volatility[2]. Section 4 considers a calibration example to FX market data, and finally Section 5 concludes.

2. MODELING FRAMEWORK

Before introducing the general model, we first recall the Jarrow and Yildirim (2003) model which can be seen as a special (degenerate) case of our model. The Jarrow and Yildirim (2003) framework for modeling inflation and real rates is based on a foreign-exchange analogy between the real of and the nominal economy. That is, the real rates are seen as interest rates in the real (foreign) economy, whereas the nominal rates represent the interest rates in the nominal (domestic) economy. The inflation index then represents the exchange rate between the nominal (domestic) and real (foreign) currency.

The general model can be seen as an extension of the models of Jarrow and Yildirim (2003), see also van Haastrecht et al. (2008). That is, instead of one-factor Hull and White (1993) models for the instantaneous nominal and real rates, we let the short rate be driven by multiple (correlated) factors. To ease the notation, we use an equivalent additive formulation for Hull-White interest rates in terms of a sum of correlated Gaussian factors plus a deterministic function, i.e., we write the model into an affine factors formulation, cf Duffie et al. (2000, 2003). The deterministic factor can be chosen as to exactly fit the term structure of the nominal or real interest rates, e.g. see Brigo and Mercurio (2006) or Pelsser (2000). If the nominal short interest rate is driven by K correlated Gaussian factors and the real short rate by M factors, the multi-factor Gaussian interest can be represented as:

$$n(t) = \varphi_n(t) + \sum_{i=1}^{K} x_n^i(t), \qquad r(t) = \varphi_r(t) + \sum_{j=1}^{M} x_r^j(t), \tag{1}$$

where $\varphi_n(t), \varphi_r(t)$ are the deterministic functions to fit the nominal and real term structure (in particular $\varphi_n(0) = n(0)$ and $\varphi_r(0) = r(0)$) and with $x_n^i(t)$, $x_r^j(t)$ the Gaussian factors which drive respectively the nominal and real rates.

The second extension included in our model is that we make the volatility σ_I stochastic. Moreover, we allow this stochastic volatility factor, which we denote by $\nu(t)$ from now on, to be correlated with the instantaneous interest rates and the inflation index. Two popular choices within the stochastic volatility literature are the models of Heston (1993) and Schöbel and Zhu (1999). In the

option can be seen as an FX-option in which (possibly deterministic) foreign interest rates represent the continuous dividend yield.

[2]We refer the reader to van Haastrecht and Pelsser (2008) for more extensive pricing results under Heston (1993) stochastic volatility.

latter the volatility is modeled as an Ornstein-Uhlenbeck process

$$d\nu(t) = \kappa\big[\psi - \nu(t)\big]dt + \tau dW_\nu(t), \quad \nu(0) = \nu_0 \tag{2}$$

with κ, ψ, σ_ν positive parameters and where $W_\nu(t)$ is a Brownian motion that is correlated with the other driving factors, especially the asset price. Note that we have a positive probability that $\nu(t)$ in (2) can become negative, which will cause the correlation between $\nu(t)$ and the other driving factors to (temporarily) change sign.

The most popular stochastic volatility model, however, is the Heston (1993) model, which mainly owns its popularity due to its analytical tractability. In the Heston model, the variance is modeled by the following Feller/CIR/square-root process

$$d\nu^2(t) = \kappa\big[\theta - \nu^2(t)\big]dt + \xi\nu(t)dW_\nu(t), \quad \nu^2(0) = \nu_0^2 \tag{3}$$

with κ, θ, ξ positive parameters and where W_ν represents again a Brownian motion that is correlated with the other model factors.

With the multi-factor Gaussian rates and with stochastic volatility like Schöbel-Zhu or Heston, we come to the following proposition for the dynamics of our model.

Proposition 2.1 *The $\mathcal{Q}_n$ dynamics of the K-factor instantaneous nominal rate $n(t)$, M-factor real rate $r(t)$ and the inflation index $I(t)$, are given by*

$$\begin{aligned}
dx_n^i(t) &= -a_n^i x_n^i(t)dt + \sigma_n^i dW_{n_i}(t), & i = 1,\dots,K, \quad (4)\\
dx_r^j(t) &= \big[-a_r^j x_r^j(t) - \rho_{I,x_r^j}\nu(t)\sigma_r^j\big]dt + \sigma_r^j dW_{r_j}(t), & j = 1,\dots,M, \quad (5)\\
dI(t) &= I(t)\big[n(t) - r(t)\big]dt + \nu(t)I(t)dW_I(t), & (6)
\end{aligned}$$

with $a_n^i, a_r^j, \sigma_n^i, \sigma_r^j$ positive parameters, $\nu(t)$ the stochastic volatility factor with dynamics given by (2) or (3), and where $(W_{n_1},\dots,W_{n_K},W_{r_1},\dots,W_{r_M},W_\nu)$ is a Brownian motion under $\mathcal{Q}^n$ with (possibly) a full correlation structure.

The multi-factor Gaussian model is still very tractable, e.g. one has analytical formulas for the prices of a zero-coupon bond.

3. PRICING AND APPLICATIONS

In this section, we will briefly discuss the main vanilla inflation, FX and equity derivatives, and discuss how these securities can be priced in a closed-form by our model. Before turning to the market-specific structures, we first consider the general pricing methodology.

3.1. Pricing

We will now discuss the general option pricing framework for inflation, FX and stock options. That is, we briefly review the framework of Carr and Madan (1999) for the pricing of European options using Fourier inversion. Afterwards, we show how this framework can be applied to value inflation, FX and stock derivatives. Under the risk-neutral measure $\mathcal{Q}$ (i.e., with the bank account as numeraire), we can write the following expression for the price $C_T(k)$ of an European option ($\omega = 1$ for a call, $\omega = -1$ for a put) maturing at time T, with strike $K = \exp(k)$, on an asset I:

$$C_T(k, \omega) = \mathbb{E}_n \left\{ e^{-\int_t^{T_i} n(u)du} \left[\omega \Big(I(T) - K \Big) \right]^+ | \mathcal{F}_t \right\}. \tag{7}$$

Hence, note that in order to price European options, we only need the probability distribution of the T-forward stock price at time T. Therefore, instead of evaluating expected discounted payoff under the risk-neutral bank account measure, we can also change the underlying probability measure to evaluate this expectation under the T-forward probability measure $\mathcal{Q}^T$ (e.g. see Geman et al. (1996)). This is equivalent to choosing the T-discount bond as numeraire. Hence, conditional on time t, we can evaluate the price of a European option ($\omega = 1$ for a call, $\omega = -1$ for a put) with strike $K = \exp(k)$ as

$$C_T(k, \omega) = P_n(t, T) \mathbb{E}_n^{\mathcal{Q}^T} \left\{ \left[\omega \Big(I_F^T(T) - K \Big) \right]^+ | \mathcal{F}_t \right\} \tag{8}$$

where $P_n(t, T)$ denotes the price of a (pure) discount bond and $I_F^T(t) := \frac{I(t)}{P_n(t,T)}$ denotes the T-forward index price. The above expression can be numerically evaluated by means of a Fourier inversion of the log-asset price characteristic function; following Carr and Madan (1999), Lewis (2001) and Lord and Kahl (2007), we can then write the call option price (7) with log strike k, in terms of the (T-forward) characteristic function ϕ_T of the T-forward log index price $z(T) := \log I_F^T(t)$. Provided that the regularity conditions for the Fourier Transformations are satisfied, i.e., $\alpha > 0$ for a call ($\omega = 1$) and $\alpha > 1$ for a put ($\omega = -1$), one can write the following for the corresponding European option price:

$$C_T(k, \omega, \alpha) = \frac{P_n(t, T)}{\pi} \int_0^\infty \mathrm{Re}\Big(e^{-(\alpha + iv)k} \psi_T(v, \omega, \alpha) \Big) dv, \tag{9}$$

with

$$\psi_T(v, \omega, \alpha) := \frac{\phi_T\Big(v - (\omega\alpha + 1)i \Big)}{(\omega\alpha + iv)(\omega\alpha + 1 + iv)}, \tag{10}$$

and where $\phi_T(u) := \mathbb{E}^{\mathcal{Q}^T} \left[\exp\Big(iuz(T) \Big) | \mathcal{F}_t \right]$ denotes the T-forward characteristic function of the log index price. Note that (9) can be efficiently evaluated, i.e., either by direct integration or by Fast Fourier Transformation, see Carr and Madan (1999), Lee (2004) or Lord and Kahl (2007). Thus, for the pricing of call and put options on some underlying asset, it suffices to know the characteristic function of the underlying. In the following sections, we derive the characteristic functions under Schöbel and Zhu (1999) stochastic volatility, whereas the case with Heston (1993) stochastic volatility is discussed in van Haastrecht and Pelsser (2008).

3.2. Schöbel-Zhu stochastic volatility

In this section we will determine the characteristic function (under the T-forward measure) of the forward log-inflation return $y(T_{i-1}, T_i)$ between times T_{i-1} and T_i. Therefore, we first need to determine the characteristic function of the T-forward log-inflation rate z_T for a general maturity T. Building on the results of van Haastrecht et al. (2008), where the characteristic function is derived for the one-factor Schöbel-Zhu-Hull-White model, we will derive its multi-factor generalization in the following subsection.

3.2.1. Characteristic function of the log-index price

Using a partial differential approach, we will now determine the characteristic function of the log index price of the dynamics (Proposition 2.1) with Schöbel and Zhu (1999) stochastic volatility. First, recall that $z(t) := \log I_F(t)$ is defined as the T-forward log-asset price; subject to the terminal boundary condition $f(T, z, \nu) = \exp\Big(iuz(T)\Big)$, the Feynman-Kac theorem implies that the expected value of $\exp\big(iuz(T)\big)$, equals the solution of the Kolmogorov backward partial differential equation for the joint probability distribution function $f(t, z, \nu)$, i.e.

$$f := f(t, z, \nu) = \mathbb{E}^{\mathcal{Q}^T}\Big[\exp\big(iuz(T)\big)\big|\mathcal{F}_t\Big]. \tag{11}$$

Thus, the solution for f equals the characteristic function of the forward asset price dynamics (starting from z at time t). To obtain the Kolmogorov backward partial differential equation for the joint probability distribution function $f = f(t, y, \nu)$, we need to take into account the covariance term $dz(t)d\nu(t)$ which equals

$$dz(t)d\nu(t) = \Big(\rho_{I\nu}\tau\nu(t) + \sum_{i=1}^{K}\rho_{x_n^i\nu}\tau\sigma_n^i B_n^i(t,T) - \sum_{j=1}^{K}\rho_{x_r^j\nu}\tau\sigma_r^j B_r^j(t,T)\Big)dt. \tag{12}$$

The model we are considering is not an affine model in $z(t)$ and $\nu(t)$, but it will be one if we enlarge the state space and include $\nu^2(t)$:

$$dz(t) = -\frac{1}{2}\nu_F^2(t)dt + \nu_F(t)dW_F^T(t), \tag{13}$$

$$d\nu(t) = \kappa\Big[\xi(t) - \nu(t)\Big]dt + \tau dW_\nu^T(t), \tag{14}$$

$$d\nu^2(t) = 2\nu(t)d\nu(t) + \tau^2 dt = 2\kappa\Big(\frac{\tau^2}{2\kappa} + \xi(t)\nu(t) - \nu^2(t)\Big)dt + 2\tau\nu(t)dW_\nu(t). \tag{15}$$

Using (13) and (12), we obtain the following partial differential equation for $f(t, z, \nu)$:

$$\begin{aligned} 0 \;=\; & f_t - \frac{1}{2}\nu_F^2(t)f_z + \kappa\big(\xi(t) - \nu(t)\big)f_\nu + \frac{1}{2}\nu_F^2(t)f_{zz} \\ & + \Big(\rho_{I\nu}\tau\nu(t) + \sum_{i=1}^{K}\rho_{x_n^i\nu}\tau\sigma_n^i B_n^i(t,T) - \sum_{j=1}^{M}\rho_{x_r^j\nu}\tau\sigma^j B_r^j(t,T)\Big)f_{z\nu} + \frac{1}{2}\tau^2 f_{\nu\nu}. \end{aligned} \tag{16}$$

Solving this partial differential equation, provides us with the characteristic function of the forward asset price dynamics (starting from z at time t). Due to the affine structure of our model, we come to the following proposition.

Proposition 3.1 *The characteristic function of domestic T-forward log inflation-rate of the model with Schöbel and Zhu (1999) stochastic volatility is given by the following closed-form solution:*

$$f(t,z,\nu) = \exp\Big[A(u,t,T) + B(u,t,T)z(t) + C(u,t,T)\nu(t) + \frac{1}{2}D(u,t,T)\nu^2(t)\Big], \tag{17}$$

where:

$$\begin{aligned}
A(u,t,T) &= -\frac{1}{2}u(i+u)V_{K,M}(t,T) \\
&\quad + \int_t^T \Bigg\{ \Bigg[\kappa\psi + (iu-1)\sum_{i=1}^{K} \rho_{x_n^i\nu}\tau\sigma_n^i B_n^i(t,T) - iu\sum_{j=1}^{M}\rho_{x_r^j\nu}\tau\sigma_r^j B_r^j(t,T)\Bigg] C(u,s,T) \\
&\qquad + \frac{1}{2}\tau^2\Big(C^2(u,s,T) + D(u,s,T)\Big)\Bigg\} ds, \\
B(u,t,T) &= B := iu, \\
C(u,t,T) &= -\frac{u\big(i+u\big)}{\gamma_1 + \gamma_2 e^{-2\gamma(T-t)}}\Bigg\{\gamma_0\Big(1 + e^{-2\gamma(T-t)}\Big) \\
&\quad + \sum_{i=1}^{K}\Big[\Big(\gamma_3^i - \gamma_4^i e^{-2\gamma(T-t)}\Big) - \Big(\gamma_5^i e^{-a_n^i(T-t)} - \gamma_6^i e^{-(2\gamma + a_n^i)(T-t)}\Big) - \gamma_7^i e^{-\gamma(T-t)}\Big] \\
&\quad - \sum_{j=1}^{M}\Big[\Big(\gamma_8^j - \gamma_9^j e^{-2\gamma(T-t)}\Big) - \Big(\gamma_{10}^j e^{-a_r^j(T-t)} - \gamma_{11}^j e^{-(2\gamma + a_r^j)(T-t)}\Big) - \gamma_{12}^j e^{-\gamma(T-t)}\Big]\Bigg\} \\
D(u,t,T) &= -u\big(i+u\big)\frac{1 - e^{-2\gamma(T-t)}}{\gamma_1 + \gamma_2 e^{-2\gamma(T-t)}}.
\end{aligned}$$

Here, $V_{K,M}(t,T)$ represents the integrated variance of the Gaussian rate processes (e.g. see van Haastrecht and Pelsser (2008)), and

$$\gamma = \sqrt{(\kappa - \rho_{I\nu}\tau B)^2 - \tau^2(B^2 - B)}, \qquad \gamma_0 = \frac{\kappa\psi}{\gamma}$$

$$\gamma_1 = \gamma + (\kappa - \rho_{I,\nu}\tau B), \qquad \gamma_2 = \gamma - (\kappa - \rho_{I,\nu}\tau B),$$

$$\gamma_3^i = \frac{\rho_{I,x_n^i}\sigma_n^i\gamma_1 + \rho_{x_n^i\nu}\sigma_n^i\tau(iu-1)}{a_n^i\gamma}, \qquad \gamma_4^i = \frac{\rho_{I,x_n^i}\sigma_n^i\gamma_2 - \rho_{x_n^i\nu}\sigma_n^i\tau(iu-1)}{a_n^i\gamma},$$

$$\gamma_5^i = \frac{\rho_{I,x_n^i}\sigma_n^i\gamma_1 + \rho_{x_n^i,\nu}\sigma_n^i\tau(iu-1)}{a_n^i(\gamma - a_n^i)}, \qquad \gamma_6^i = \frac{\rho_{I,x_n^i}\sigma_n^i\gamma_2 - \rho_{x_n^i,\nu}\sigma_n^i\tau(iu-1)}{a_n^i(\gamma + a_n^i)},$$

$$\gamma_8^j = \frac{\rho_{I,x_r^j}\sigma_r^j\gamma_1 + \rho_{x_r^j,\nu}\sigma_r^j\tau B}{a_r^j\gamma}, \qquad \gamma_9^j = \frac{\rho_{I,x_r^j}\sigma_r^j\gamma_2 - \rho_{x_r^j,\nu}\sigma_r^j\tau B}{a_r^j\gamma},$$

$$\gamma_{10}^j = \frac{\rho_{I,x_r^j}\sigma_r^j\gamma_1 + \rho_{x_r^j,\nu}\sigma_r^j\tau B}{a_r^j(\gamma - a_r^j)}, \qquad \gamma_{11} = \frac{\rho_{I,x_r^j}\sigma_r^j\gamma_2 - \rho_{x_r^j,\nu}\sigma_r^j\tau B}{a_r^j(\gamma + a_r^j)},$$

$$\gamma_7^i = (\gamma_3^i - \gamma_4^i) - (\gamma_5^i - \gamma_6^i), \qquad \gamma_{12}^j = (\gamma_8^j - \gamma_9^j) - (\gamma_{10}^j - \gamma_{11}^j).$$

Proof. See van Haastrecht and Pelsser (2008). ■

3.2.2. Characteristic function of log index return

Recently, the pricing of forward starting options attracted the attention of practitioners as well as from academia, see e.g. Lucić (2003), Hong (2004), van Haastrecht et al. (2008), and in an inflation context Mercurio and Moreni (2006) and Kruse (2007). In this section we will consider the pricing of forward starting options like inflation caplets within the general model setup combined with Schöbel-Zhu volatility. In particular, using the framework of Carr and Madan (1999), as described in section 3.1, it suffices to know the characteristic function of the following log-inflation index return under the T_i-forward measure:

$$y(T_{i-1},T_i) := \log\Big(\frac{I(T_i)}{I(T_{i-1})}\Big) = \log I(T_i) - \log I(T_{i-1}). \tag{18}$$

First of all, recalling that $I(t) := I_F(t)\frac{P_n(t,T_i)}{P_r(t,T_i)}$ and $z(t) := \log I_F(t)$, we can also express this return in terms of the T_i-forward log inflation rate:

$$y(T_{i-1},T_i) = z(T_i) - z(T_{i-1}) - \log P_n(T_{i-1},T_i) + \log P_r(T_{i-1},T_i). \tag{19}$$

We then want to derive the characteristic function $\phi_{T_{i-1},T_i}(u)$ of the log-inflation index return $y(T_{i-1},T_i)$ under the T_i forward measure, i.e.

$$\phi_{T_{i-1},T_i}(u) := \mathbb{E}^{\mathcal{Q}^T}\Big[\exp\Big(iu\big(y(T_{i-1},T_i)\big)\Big)\big|\mathcal{F}_t\Big]. \tag{20}$$

To this end, define

$$\Lambda := \exp\Bigg(iu\Big[z(T_i) - z(T_{i-1}) - \log P_n(T_{i-1},T_i) + \log P_r(T_{i-1},T_i)\Big]\Bigg). \tag{21}$$

Using the tower law for conditional expectations and the (conditional) characteristic function of our model (17), we obtain the following expression for the characteristic function of the (forward) log-return:

$$\begin{aligned}
\phi_{T_{i-1},T_i}(u) &= \mathbb{E}_n^{T_i}\Big\{\Lambda\Big|\mathcal{F}_t\Big\} = \mathbb{E}_n^{T_i}\Big\{\mathbb{E}_n^{T_i}\Big[\Lambda\Big|\mathcal{F}_{T_{i-1}}\Big]\Big|\mathcal{F}_t\Big\} \\
&= \mathbb{E}_n^{T_i}\Big\{\exp\Big(iu\Big[-z(T_{i-1}) - \log P_n(T_{i-1},T_i) + \log P_r(T_{i-1},T_i)\Big]\Big) \\
&\qquad \cdot\mathbb{E}_n^{T_i}\Big[\exp\big[iuz(T_i)\big]\big|\mathcal{F}_{T_{i-1}}\Big]\Big|\mathcal{F}_t\Big\} \\
&= \exp\Big(iu\Big[A_r(T_{i-1},T_i) - A_n(T_{i-1},T_i)\Big] + A(u,T_{i-1},T_i)\Big) \\
&\quad \cdot\mathbb{E}_n^{T_i}\Big\{\exp\Big(iu\Big[\sum_{k=1}^{K} B_n^k(T_{i-1},T_i)x_n^k(T_{i-1}) - \sum_{j=1}^{M} B_r^j(T_{i-1},T_i)x_r^j(T_{i-1})\Big] \\
&\qquad + C(u,T_{i-1},T_i)\nu(T_{i-1}) + \frac{1}{2}D(u,T_{i-1},T_i)\nu^2(T_{i-1})\Big)\Big|\mathcal{F}_t\Big\}.
\end{aligned} \tag{22}$$

Because this expectation only depends on the (correlated) Gaussian variates $x_n^k(T_{i-1})$, $x_r^j(T_{i-1})$, $\nu(T_{i-1})$ and $\int_t^{T_{i-1}} \nu(u)du$ (see van Haastrecht and Pelsser (2008)), one finds that the characteristic function (22) is of the following Gaussian-quadratic form:

$$\mathbb{E}_n^{T_i}\left\{\exp\left[a_0 + \boldsymbol{a}'\boldsymbol{X} + \boldsymbol{X}'\boldsymbol{B}\boldsymbol{X}\right])\right\}, \tag{23}$$

with a_0 a constant, $\boldsymbol{a}'$ a row-vector, $\boldsymbol{B}$ a matrix, and where $\boldsymbol{X}$ follows a multivariate standard normal distribution with correlation matrix $\boldsymbol{S}$. Therefore, using standard theory on Gaussian-quadratic forms (e.g. see Glasserman (2003) or Feuerverger and Wong (2000)), one can now easily find an explicit solution for the forward characteristic function by evaluating the above Gaussian-quadratic form, see van Haastrecht and Pelsser (2008).

4. CALIBRATION EXAMPLE

In this section, we consider two applications (i.e., one with Schöbel and Zhu (1999) and one with Heston (1993) stochastic volatility) in which we calibrate our model to FX (option) market data. The example explicitly takes into account the pronounced long-term FX implied volatility skew/smile that is present in the markets. After calibration, we compare and analyze the results.

4.1. FX market

We will test our model by calibrating it to FX option market data. To this end, we consider the same vanilla FX data as in Piterbarg (2005) where the author uses this set for the calibration of his local volatility model. In an elegant paper, Piterbarg (2005) concludes that for the pricing and managing of exotic FX derivatives (i.e., PRDCs), it is essential to take into account the FX implied volatility skew/smile. Hence, though FX model setups may differ –i.e., local volatility in Piterbarg (2005) and Heston (1993), stochastic volatility with independent stochastic interest rate drivers in Andreasen (2006), and our stochastic volatility model with multi-factor Gaussian rates and Heston (1993) or Schöbel and Zhu (1999) volatility under a full correlation structure– all these models share the essential feature of explicitly accounting for the FX skew/smile.

For the calibration results of our model we consider the same interest rate and correlation parameters as in Piterbarg (2005), see van Haastrecht and Pelsser (2008).

4.2. Calibration results

We calibrate the models (Proposition 2.1) with Schöbel and Zhu (1999) and Heston (1993) stochastic volatility to the various maturities by minimizing the differences between model and market implied volatilities using a local optimization method. The results are plotted in the graphs below.

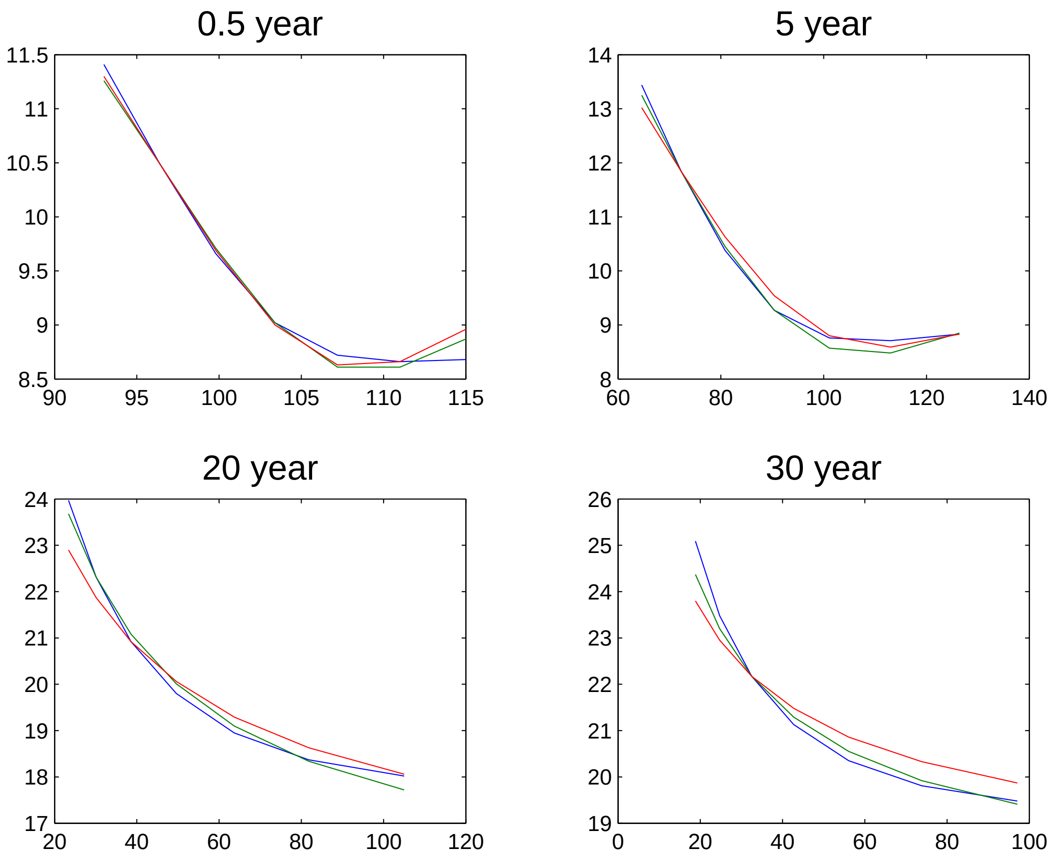

Figure 1: Calibration results for the model with Schöbel-Zhu and Heston stochastic volatility. For the maturities 0.5, 5, 20 and 30 the implied volatilities (vertical axis) are plotted against the corresponding strikes (horizontal axis). The market data is represented in blue, the model with Schöbel-Zhu volatility in red and the model with Heston volatility in green.

We first consider the model (Proposition 2.1) with Schöbel and Zhu (1999) stochastic volatility. The model produces a good fit to the market, as can be seen from Figure 1, with differences smaller than 0.50% in most points and with a good fit around the at-the-money-forward volatilities and the slope of the volatility skews for each maturity. The model produces similar calibration results as the models of Piterbarg (2005) and Andreasen (2006). The low-strike (in-the-money call) options are underestimated by the model, which seems to cause slight difficulties in fitting the tails of the implied volatility structure, suggesting the addition of an extra factor, e.g. a trivial extension including Poisson type jumps. Nonetheless, the smiles produced by the model are much closer to the market than a log-normal model would indicate, in particular for in- and out-the-money options.

Secondly, we consider the model (Proposition 2.1) with Heston (1993) stochastic volatility. For simplicity we have considered uncorrelated stochastic volatility, as we can then directly price the required FX options in closed form. Nonetheless, the calibration results to call option prices should be very similar as it is shown in Antonov et al. (2008). The parameters of the general model can often be projected onto parameters of the uncorrelated model, while to a large extent preserving option prices and model characteristics. The calibration results can be found in Figure 1. It can be seen that the model again produces a very good fit to the market, with differences now even smaller than 0.30% in most points and with excellent fits across moneyness and maturities. It seems that Heston (1993) model is slightly better in fitting extreme/convex FX skew: in a way it is able to capture both the volatility part of the at-the-money prices, as well as the extremes of the in- and out-the-money prices. Alternatively, one can argue that the addition of an extra factor is still needed for the pricing of certain exotic options (e.g. see Fouque et al. (2000)), a discussion which is, however, beyond the scope of this article.

It is shown in Piterbarg (2005) and Andreasen (2006), that it is of crucial importance to take the FX skew into account for the pricing and managing of exotic FX structures like PRDCs (power reverse dual contracts) or cliquets. Therefore, since the skews/smiles generated by our stochastic volatility models are much closer to the market than the skews/smiles produced by a log-normal model, we can conclude that our stochastic volatility models (Proposition 2.1) are better suited to price and manage these exotic FX structures. Finally, though the models of Piterbarg (2005) and Andreasen (2006) account for the FX skew, our model stands out as we model stochastic volatility (versus local volatility used in Piterbarg (2005)) and stochastic interest rates, whilst we allow all driving model factors to be instantaneously correlated with each other (versus independent Gaussian rates used in Andreasen (2006)). Having this flexibility yields a realistic model, which is of practical importance for the pricing and hedging of options with a long-term FX exposure.

5. CONCLUSION

We have introduced a generic model incorporating stochastic interest rates and stochastic volatility under a full correlation structure of all driving model factors, with closed-form pricing formulas for vanilla options and which is able to incorporate the markets implied volatility structures. Due to the flexibility to correlate the underlying FX/Inflation/Stock-index with both the stochastic volatility and the stochastic interest rates, our approach yields a realistic model, which is of practical importance for the pricing and hedging of options with a long-term exposure. Furthermore, closed-form pricing of vanilla FX, Inflation and stock options is a big advantage for the calibration (and sensitivity analysis) of the model. Using Fourier methods, we have shown how vanilla call/put options, forward starting options, year-on-year inflation-indexed swaps and inflation-indexed caps/floors can be valued in closed-form. Hereby, it must be noted that our model can cover Poisson type jumps with a trivial extension. Under Schöbel and Zhu (1999) stochastic volatility, using its affine properties, we were able to derive the corresponding characteristic functions in closed-form. Under Heston (1993) stochastic volatility, the characteristic functions can only be derived explicitly under special zero correlation assumptions. Nonetheless, as an alternative, one can use this result as a control variate for the general model, see van Haastrecht and Pelsser (2008). Our model can be used for multi-asset purposes (e.g. interest rates, FX, inflation, equity, commodities) and is fast enough for the real life risk management of big portfolios of such products. We think it is particularly suitable for the pricing and hedging of long-dated multi-currency structures (e.g. hybrid TARN options, variable annuities, inflation LPI options and PRDC FX swaps) which are sensitive to both future interest rates evolutions as well as movements from the underlying index and/or corresponding volatility smiles.

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EXTENDED ABSTRACTS
POSTER SESSION

VANNA-VOLGA METHODS APPLIED TO FX DERIVATIVES: FROM THEORY TO MARKET PRACTICE

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The Foreign Exchange (FX) options market is the largest and most liquid market of options in the world. First-generation exotics (touch-like options and vanillas with barriers) are becoming so popular products in FX that it makes imperative for any pricing system to provide a fast and accurate mark-to-market pricing for this family of products. Although using the Black-Scholes model it is possible to derive analytical prices for barrier and touch options, this model is unfortunately based on several unrealistic assumptions that render the price inaccurate. In particular, the Black-Scholes model assumes that the foreign/domestic interest rates and the FX-spot volatility remain constant throughout the lifetime of the option. This is clearly wrong as these quantities change continuously. More realistic models should assume that the foreign/domestic interest rates and the FX spot volatility follow stochastic processes that are coupled to the one of the spot. On the other hand, for short-dated options (typically less than 1 year), assuming constant interest rates does normally not lead to significant errors. In this article we assume constant interest rates throughout.

Stochastic volatility models are unfortunately computationally demanding and in most cases require a delicate calibration procedure in order to find the value of parameters that allow the model to reproduce the market dynamics. This has led to alternative ad-hoc pricing techniques which give fast results and are simpler to implement. One such approach is the Vanna-Volga (VV) method which, in a nutshell, consists in adding an analytically derived correction to the Black-Scholes price of the instrument. To do that, the method uses a small number of market quotes for liquid instruments (typically At-The-Money options, Risk Reversal and Butterfly strategies) and constructs an over-hedge which zeroes out the Black-Scholes Vega, Vanna and Volga of the option. The choice of this set of Greeks is linked to the fact that they all offer a measure of the option's sensitivity with respect to the FX-spot volatility, and therefore the constructed over-hedge aims to take the smile effect into account. Although the real interest in the approach lies in its effectiveness at producing reasonable estimates of the market prices of first-generation exotics, a sound theoretical justification can only be derived in the case of vanilla options.

In this article we give two interpretations of the VV method to price vanilla options. We show that the Vanna-Volga price of a vanilla can be seen as a Taylor expansion of the exact vanilla price (with the exact market volatility directly plugged into the B&S formula), in the vicinity of the At-The-Money-Forward strike. Then we justify the Vanna-Volga method as an approximation to a stochastic volatility model. Next we review two well known VV variations used to price exotic options. The first one consists in weighting the Vanna-Volga correction by some function of the survival probability, the second one is based on the expected first hitting time argument.

In order to assess the ability of the two above variations of the Vanna-Volga method to reflect market prices, we compared them to a large collection of market indicative prices collected from trading platforms of the three major FX-option market-makers. In addition to comparing the VV models with market prices, we also compared them to a local-volatility model (Dupire), and a stochastic volatility model (Heston) calibrated to the smile at maturity and the ATM volatility term structure. It appears that for FX markets characterized by a dominantly skewed implied volatility, the Dupire model is qualitatively similar to the two Vanna-Volga models (while the Heston one follows a different trend) and, inversely, in a FX market characterized by a mild skew, the Heston model aligns towards the Vanna-Volga prices while the Dupire one rebels away. However, Heston and Dupire models (calibrated to the vanilla market) seem less performing (for this dataset) when compared to the Vanna-Volga method after it has been calibrated. This confirms that calibrating a stochastic model to the vanilla market is by no mean a guarantee that exotic options will be priced correctly, as a vanilla market carries no information about the smile dynamics.

In the literature, there is no agreed consensus regarding whether the survival probability or the expected first hitting time is a better candidate for adjusting the Vanna-Volga recipe to price exotics. Based on some empirical observations, it is suggested that one uses some function of the survival probability. Other market beliefs however favor using a function of the first exit time. Other adjustment possibilities are also suggested, depending on the type of option at hand. But there exists no mathematical argument to justify these choices. In this article we discuss a more systematic procedure to calibrate the Vanna-Volga model. This calibration allows to reproduce market prices in a better way and ensures no-arbitrage opportunities.

The FX derivatives community, perhaps more than any other asset class, lives on a complex structure of quote conventions. Naturally, a wrong interpretation of the input market data cannot lead to the correct results. To this end, we have presented some relevant FX conventions regarding smile quotes and we have tested the robustness of the Vanna-Volga method against the input data. It appears that the values of Vanna and Volga provide a good indication of whether we are in a region of parameters where the method is very sensitive with respect to its input.

AN ANALYSIS OF THE UNDERWRITING CYCLE FOR NON-LIFE INSURANCE COMPANIES

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The European Solvency II project (see CEIOPS (2007)) introduces Solvency Capital Requirements that capture the overall risk profile of insurance companies, see also Sandström (2005). In this framework there is a growing need to develop so-called internal risk models to get accurate estimates of liabilities. In the context of non-life insurance, it is crucial to correctly assess risk from different sources, such as underwriting risk with particular reference to premium, reserving and catastrophe risks.

Especially the underwriting cycle providing additional volatility, can lead to considerable capital requirement (see Meyers (2007)). In fact, in adverse development of financial positions, company's management typically tends to raise premium rates or reinsurance, whereas otherwise it increases dividends or reduces premium rates. See Choi et al. (2002), Cummins and Danzon (1997), Derien (2008), Feldblum (2001), Gron (1994), Higgins and Thistle (2000) for different approaches and extensive analysis.

A correct analysis of this phenomena is also significant to understand the evolution of the reserving cycle, which is often correlated, with a lag period, with the underwriting cycle. Indeed it has been ascertained the tendency of insurers to over-estimate technical reserves during the hard part of the cycle, when loss ratios are low, and under-estimate these reserves in the opposite case.

The aim of this paper is to correctly model the underwriting cycle for non-life insurance companies.

The basic model is derived from Collective Risk Theory. Starting from the idea of Pentikainen et al. (1989) and Daykin et al. (1994), in this paper a dynamic control policy is defined to specify the relationship between solvency ratio $u(t)$ and safety loadings $\lambda(t)$, in order to model the Underwriting Cycle. In particular a simplified formula of $\lambda(t)$ is considered; it assumes the form of a one dimensional piecewise linear map in the state variable $u(t)$.

Firstly, a deterministic version of this map is analyzed, where aggregate losses x are simply regarded as a parameter. In this case, a local and a global analysis of $u(t)$ is performed, showing that the long run equilibrium of the solvency ratio $u(t)$ can present jump discontinuities as the main parameters of the model vary. Technically these discontinuities are a consequence of a particular double "Border-collision" bifurcation of the underlying map (see Budd et al. (2008) for details), and are related to the crossing of the trajectory of $u(t)$ into regions where the definition of the map changes. Stochastic assessments of $u(t)$ conclude the work.

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IMPLIED LÉVY VOLATILITY

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We introduce the concept of implied Lévy volatility, hereby extending the intuitive Black-Scholes implied volatility into a more general context. More precisely, Lévy implied time and space volatility are introduced and a study of the shape of implied Lévy volatilities is made.

Model performance is studied by analyzing delta-hedging strategies for the Normal Inverse Gaussian and the Meixner model, both qualitatively and on historical time-series of the S&P500.

1. IMPLIED BLACK-SCHOLES VOLATILITY

The Black-Scholes model uses geometric Brownian motion to model the diffusion part of the log-return process:

$$S_t = S_0 \exp((r - q - \sigma^2/2)t + \sigma W_t), \;\; t \geq 0.$$

The *Black-Scholes implied volatility* is the volatility $\sigma = \sigma(K, T)$ such that the model and market option prices coincide.

The concept of implied volatility under the Black-Scholes model is one of the key points of its success and its widespread use. In fact it gives another, more convenient and robust, way of quoting plain vanilla European option prices. Over the years, option traders have developed an intuition in this quantity. As it turns out, this model parameter depends on the characteristics of the contract. More precisely, it depends on the strike price and the remaining lifetime of the option. The precise functional form is called the volatility surface and follows its own dynamics in the market. This model parameter needs to be adjusted separately for each individual contract given the inadequacy of the underlying Black-Scholes model.

By analyzing empirical historical data, it is not hard to see that stock returns tend to be more skewed and have fatter tails than those the normal distribution can provide. Here a similar concept is developed but now under a Lévy framework and therefore based on distributions that match more closely historical returns.

2. IMPLIED LÉVY VOLATILITY MODELS

The Lévy models are obtained by replacing the Wiener distribution modeling the diffusion part of the log-return process by a more empirically founded Lévy distribution. The Lévy space volatility model will arise by multiplying volatility with the underlying Lévy process, whereas the Lévy time volatility model will arise by multiplying volatility squared with time:

$$S_t = S_0 \exp((r - q + \omega_{\text{space}})t + \sigma_{\text{space}} X_t), \quad t \geq 0,$$

and

$$S_t = S_0 \exp((r - q + \omega_{\text{time}}\sigma^2_{\text{time}})t + X_{\sigma^2_{\text{time}} t}), \quad t \geq 0,$$

where $E[X_1] = 0$, $\text{Var}[X_1] = 1$, $\omega_{\text{space}} = -\log(\phi(-\sigma\mathrm{i}))$ and $\omega_{\text{time}} = -\log(\phi(-\mathrm{i}))$ where ϕ represents the characteristic function of X_1: $\phi(u) = E\left[\exp(iuX_1)\right]$. The volatility parameter $\sigma_{\text{space}} = \sigma_{\text{space}}(K,T)$ and $\sigma_{\text{time}} = \sigma_{\text{time}}(K,T)$ needed to match the model price with a given market price is called the *implied Lévy space volatility* and the *implied Lévy time volatility*, respectively.

3. STUDYING IMPLIED LÉVY VOLATILITY SHAPES

By switching from the Black-Scholes world to the Lévy world, we introduce additional degrees of freedom which can be used in order to minimize the curvature of the volatility surface. We look how Black-Scholes curves are translated into implied Lévy volatility curves and vice versa. It is shown that any smiling or smirking Black-Scholes volatility curve can be transformed into a flatter Lévy volatility curves under a well chosen parameter set. This gives some evidence to the fact that the implied Lévy models could lead to flatter volatility curve for more practical datasets. Hence, implied Lévy volatility models can be of a particular interest for practitioners facing the problem of pricing barrier options since for the Black-Scholes model, it is not clear which volatility one should use (the one of the barrier or the one of the strike).

4. IMPROVING THE DELTA HEDGE

Model performance is studied by analyzing delta-hedging strategies for short term ATM vanilla options under the Normal Inverse Gaussian and the Meixner model, both qualitatively and on historical time-series of the S&P500. The Lévy degrees of freedom can thus be determined such that the absolute value of the mean and the square root of the variance of the daily hedging error are minimized. It is shown that using the historical optimal parameters leads to a significant reduction of the variance of the hedging error (amounting to more than 50 percents), which is particularly attractive for option hedging.

A GEOSTATISTICAL APPROACH FOR DYNAMIC LIFE TABLES. THE EFFECT OF MORTALITY ON REMAINING LIFETIME AND ANNUITIES[1]

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Our contribution consists of applying geostatistical techniques for estimating the dependence structure of the mortality data in a dynamic life table and for predicting purposes. We compare the performance of this new approach with the classic Lee-Carter model with one and two terms and with an enlarged version including the influence of the year of birth (cohort). Additionally, we obtain bootstrap confidence intervals for predicted q_{xt} resulting from applying both methodologies, and we study their influence on the predictions of e_{65t} and a_{65t}.

1. A BRIEF DESCRIPTION

Static life tables do not take into account the fact that mortality progresses over the years. The concept of a dynamic life table seeks to solve this problem by jointly analyzing mortality data corresponding to a series of consecutive years. This approach allows the calendar effect influence on mortality to be studied. A sample of the models developed for graduating dynamic tables can be found in Tabeau et al. (2001), Pitacco (2004), Wong-Fupuy and Haberman (2004), Debón et al. (2006). Most of them adapt traditional laws to the new situation and all them share a common hypothesis: they consider the observed measures of mortality as independent across ages and over time. As Booth et al. (2002) point out, it is difficult to hold such a hypothesis when looking at the graph of the residuals obtained after the adjustment with any of these models.

Geostatistical techniques were designed for the analysis of data which were very far from what a dynamic table represents (Matheron 1975). This distance is only apparent as a dynamic table can actually be considered as a set of data over a rectangular grid equally spaced both vertically, for age, and horizontally, for year. The aim of Geostatistics is to model the dependence structure among neighbours, which requires defining a neighbourhood relationship as well as a distance.

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They are straightforward in the case of spatial data but also possible in other kind of data. The analysis of sudden infant death syndrome (SIDS) in North Carolina in Cressie (1993), as well as the analysis using spatial techniques of the 1970 US Draft Lottery (Mateu et al. 2004), support this assessment. Moreover, as in previous studies, we will show that these methods provide better solutions than the classical methods since they simultaneously take into account the effect of age and time, while the others treat both effects separately.

This works introduces the original Lee-Carter model with one and two time terms and the Lee-Carter age-period-cohort model, derived from the original one but adding a second term for collecting the influence of cohort over mortality, and also provides an introduction to geostatistical methodology, including a brief description of the median polish algorithm that will be used for estimating the deterministic trend of the geostatistic model proposed, namely a Gneiting model. The bootstrapping techniques used for obtaining confidence intervals are briefly presented. The seven models, three Lee-Carter and four median polish models, are applied for modeling the mortality data in Spain for the period 1980-2003 and a range of age from 0 to 99. As the crude estimates of $\dot{q}_{x,2004}$ and $\dot{q}_{x,2005}$ are known, the predictions for $\hat{q}_{x,2004}$ and $\hat{q}_{x,2005}$ are obtained in order to measure the goodness of prediction for each model. Confidence intervals for the prediction of residual life expectancy, e_{65t}, and the annual rates, a_{65t}, for $t = 2004, \ldots, 2023$, are calculated. Finally, we discuss the conclusions that can be drawn from comparing the results for different models.

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PRICING AND HEDGING ASIAN BASKET SPREAD OPTIONS

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We consider a security market consisting of m risky assets and a risk-less asset with rate of return r. We assume that under the risk-neutral measure Q the price process dynamics are given by

$$dS_{jt} = rS_{jt}dt + \sigma_j S_{jt} dB_{jt},$$

where $\{B_{jt} : t \geq 0\}$ is a standard Brownian motion associated with asset j. Further we assume that the asset prices are correlated according to

$$\text{cov}(B_{jt_v}, B_{it_s}) = \rho_{ji}\text{min}(t_v, t_s).$$

Given the above dynamics, the price of the j^{th} asset at time t_i equals

$$S_{jt_i} = S_j(0)e^{(r-\frac{\sigma_j^2}{2})t_i + \sigma_j B_{jt_i}}.$$

With this in hand we can define an Asian basket spread as

$$V = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{m}\varepsilon_j a_j S_{jt_i},$$

where a_j is the weight given to asset j and ε_j its sign in the spread. We assume that $\varepsilon_j = 1$ for $j = 1, \ldots, p$, $\varepsilon_j = -1$ for $j = p+1, \ldots, m$, where p is an integer such that $1 \leq p \leq m-1$ and $t_0 < t_1 < t_2 < \cdots < t_n = T$. The price of an Asian basket spread with exercise price K at $t_0 = 0$ can be defined as

$$e^{-rT}E_Q(V-K)_+, \tag{1}$$

with $(x)_+ = \max(x, 0)$ and where E_Q represents the expectation taken with respect to the risk-neutral measure Q.

Examples of such contracts can be found in the energy markets. The basket spread part may for example be used to cover refinement margin (crack spread) or the cost of converting fuel into energy (spark spread). While the Asian part (the temporal average) avoids the problem common to the European options, namely that speculators can increase gain from the option by manipulating the price of the assets near maturity.

Since the density function of a sum of non-independent log-normal random variables has no closed-form representation, there is no closed-form solution for the price of a security whenever $m > 1$ or $n > 1$ within the Black and Scholes framework. Therefore one has to use an approximation method when valuating such a security. It is always possible to use Monte Carlo techniques to get an approximation of the price. However, such techniques are rather time-consuming. Furthermore financial institutions also need approximations of the hedge parameters in order to control the risk, which further increases the computation time. This explains why the research for a closed-form approximation has become an active area.

In the first part of the paper we derive approximation formulae for expression (1) using comonotonic bounds. Traditionally comonotonic approximations were used to derive bounds for the prices of Asian and basket options. To the best of our knowledge this is the first time that comonotonicity is used to derive approximations to the price of basket spread and Asian basket spread options. Using the theory of comonotonicity we derive four different approximations: the upper, the improved upper, the lower and the intermediary bound.

In the second part of the paper we try to approximate the security price with the help of moment matching techniques. In this paper we improve two well known moment matching approximations, namely the hybrid moment matching and the shifted log-normal approximation.

We find that the comonotonic improved upper bound offers a good approximation of the price of spread options. We try several approximation methods for basket spread options and find that a combination of hybrid moment matching combined with the comonotonic improved upper bound and shifted log-normal moment matching seems to work best. Finally, for the approximation of Asian basket spread options we recommend the use of hybrid moment matching combined with the comonotonic improved upper bound.

We explain which method should be used depending on the basket characteristics. We also provide closed-form formulae for the Greeks of our selected approximation techniques. Finally, we explain how our results can be adapted in order to deal with options written in foreign currency (compo and quanto options).

RISK INDIFFERENCE PRICING AND BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS

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In a complete market, it is well-known that there is a unique linear arbitrage-free pricing rule for a contract with payoff G at maturity. This price is the expectation of the discounted payoff G with respect to the so-called equivalent martingale measure.

Assuming more realistically that markets are incomplete, the situation becomes more complicated. There are infinitely many equivalent martingale measures which might be used in the pricing formula. The no-arbitrage assumption provides a set of equivalent martingale measures and an interval of arbitrage-free prices. There is no exact replication recipe to provide a unique price. One possibility for a trader is to charge a super-replication (super-hedging) price for selling an option so that he can trade to eliminate all risks. However, this price is usually forbiddingly high. For example, the super-replication price for a European option is the trivial upper bound of the no-arbitrage interval. In the most common example of a call option, the super-hedging strategy is to buy and hold the underlying and therefore the price of the call is equal to the initial stock price, which is excessively expensive and wipes out any advantage that an insurance contract should have, i.e., intrinsic leverage.

In other words, the gap between upper (seller) and lower (buyer) hedging price is too wide. Since super-hedging is not a realistic solution under such circumstance, the trader is restricted to charging a reasonable price, finding a partial hedging strategy according to some optimality criterion, and bearing some risks in the end.

There are two major approaches that have been developed for pricing and hedging in incomplete markets. One is to pick a specific martingale measure for pricing according to some optimality criterion, and the other is utility-based derivative pricing. In brief, the pitfall of the first method is that it does not provide financially reasonable hedging strategies, whilst the second one requires the trader to explicitly write down his utility function, which is quite unusual in practice.

In this work, we investigate a pricing formula in incomplete markets based on the risk indifference principle. We replace the criterion of maximizing utility by minimizing risk exposure because the latter is more often used in practice and because it is a natural extension to the idea of pricing and hedging in complete markets. Because the theory of pricing fundamentally relies on hedging risk, it is interesting to have a pricing principle directly based on risk. Moreover, risk indifference pricing preserves the advantage of utility indifference pricing, mainly, its economic justification, while avoiding its limitations, essentially, the lack of explicit calculations outside exponential utility models. The idea is that the trader buys or sells the option for an amount such that with active hedging his risk exposure will not increase at expiration.

Using a dual characterization of risk measures, we show that a risk indifference pricing problem reduces to two (zero-sum) stochastic differential games. Then we solve these stochastic differential games by means of backward stochastic differential equation (BSDE) theory and find an explicit formula for the risk indifference price. We follow the spirit of Øksendal and Sulem (Risk indifference pricing in jump diffusion markets, to appear in Mathematical Finance 2009) who study a similar risk indifference pricing problem. Importantly, our (stochastic analysis) approach does not impose Markovian assumptions on the coefficients – it is a well-known benefit of BSDE over PDE techniques – and it encompasses the case of dynamic (time-consistent) risk measures.

We define the seller's dynamic risk indifference price p_t^{risk} as the payment that makes the risk involved for the seller of a contract equal to the risk involved if the contract is not sold, at all times. Assume that S is the stock price process, g is the payoff function of the (path-dependent) option contract and λ and h are functions determined by the penalty function (of the dual representation) of the risk measure. Then, for an optimal θ_t^*, we prove that the risk indifference price p_t^{risk} satisfies

$$p_t^{risk} = \frac{R_t^G - R_t^0}{K_{\theta_t^*}}$$

where K_θ is the process defining the set of equivalent martingale measures and (R_t^G, Z_t) is the solution of the following BSDE:

$$R_t^G = g(S) - h(Y) - \int_t^T \lambda(t, y, \theta_t^*)dt - \int_t^T Z_t dW_t,$$

Y being a vector process that includes S and K_θ. Similarly, (R_t^0, Z_t) is the solution of the above BSDE with terminal value equal to $-h(Y)$.

Our results show that the choice of the penalty function is crucial for it determines the shape of the risk measure, the optimal martingale measure and the size of the price interval. Indeed, the difference between R_t^G and R_t^0 depends essentially on the functions λ and h, and the optimal θ_t^* depends on λ as well.

SYNDICATED SECURED LOAN DERIVATIVES: MODELLING OF LCDS AND PRICING OF LCDX TRANCHES

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Although the market is busy today working on the bullet LCDS contract to remove the cancellation feature from syndicated secured loan derivatives, in their current form LCDSs and LCDX tranches are still exposed to the cancellation risk. Until recently, in lack of a proper modelling framework, market practitioners neglected the cancellation risk and they priced and hedged these products as simple CDSs and CDO tranches.

However, as we show in this paper, it is more than important to take into account the cancellation risk when marking-to-market and hedging syndicated secured loan derivatives. This is especially true in the current market situation. We present easy and robust techniques to model the cancellation feature. We focus on modelling and pricing of syndicated secured loan derivatives. This incorporates the joint modelling of CDS and LCDS spreads (single-name derivatives) as well as the pricing and hedging of LCDX tranches (multi-name derivatives).

1. SINGLE-NAME DERIVATIVES

The major difficulty when modelling syndicated secured loan derivatives is to retrieve the implied cancellation probabilities from traded market instruments. In Dobránszky (2008) we show that in spite of some market believes these implied cancellation probabilities can hardly be derived solely from CDS and LCDS spreads. This happens because the fair spread of an LCDS is not significantly sensitive to the cancellation rate, which amortizes the credit leg and the fee leg closely equally. However, the risky annuity used for marking-to-market LCDS contracts is more than sensitive to the cancellation rate. Therefore, even if the implied cancellation probabilities can hardly be derived solely from CDS and LCDS spreads, an assumption about the cancellation rate is essential for marking-to-market LCDS contracts. For the purpose of the analysis a new reduced-form model is introduced, which copes with correlated default and cancellation intensities as well as with correlated default intensity and stochastic recovery rates.

2. MULTI-NAME DERIVATIVES

In Dobránszky and Schoutens (2008), a new multi-name LCDS model is introduced, which can cope with the problem of implied cancellation probabilities and may help in hedging LCDX tranches. In this paper, the class of one-factor models is extended with cancellation feature. We analyse various Lévy copulas applied for pricing LCDX tranches. Numerical experiments are presented. In Dobránszky and Schoutens (2009), we show how the cancellation rates can be calibrated from super senior tranche quotes. Furthermore, we carry out a historical analysis to show, that the model extended with the cancellation feature produces more stable and flatter base correlation curves, as well as that the index hedge implied by the model results in smaller standard deviation of the daily PnL comparing to the deviation implied by the hedge disregarding from cancellation. We analyse the impact of cancellation on the marking-to-market and we address the problem of hedging the cancellation risk.

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PORTFOLIO INSURANCE, IS IT TRUE THAT COMPLEXITY LEADS TO BETTER PERFORMANCES?

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In the last twenty five years, a considerable number of Portfolio Insurance methods have emerged in Financial Markets. Portfolio Insurance methods fit a group of techniques of different complexity degrees. Since Leland (1980) and Rubinstein and Leland (1981), this strategy, deeply rooted in the options valuation theory, has been developed in the sense of guaranteeing the same goals with the simplest techniques. The importance of Portfolio Insurance, as a hedging strategy, arises from the asymmetric risk preferences of investors. Portfolio Insurance allows investors to limit their downside risk, while retaining exposure to higher returns.

The aim of this article is to discover if it is necessary to implement the more complex Portfolio Insurance techniques or if the simplest ones provide good performances. To achieve our purpose, we apply three Portfolio Insurance strategies: the Stop-Loss strategy, the CPPI (Constant Proportion Portfolio Insurance) and the OBPI (Option Based Portfolio Insurance).

Portfolio Insurance techniques have their roots in the Black and Scholes option pricing theory. In Black and Scholes (1973) a non-arbitrage argument is used to derive the model equation. This arbitrage argument can also be used to synthetically create options. OBPI was the first proposed strategy of Portfolio Insurance, see Leland (1980) and Rubinstein and Leland (1981). OBPI uses the Black and Scholes options valuation model to create a continuously adjusted synthetic European put. The Stop Loss strategy, in its most basic version, settles on a simple proposition: a floor (F) at a maturity 1 is fixed, which is the minimum value allowed to the portfolio. The initial investment is fully applied in the stock. The floor present value is periodically compared with the portfolio value. Two different situations may occur: *i*) If the portfolio value, at time t, is higher than the floor present value, $P_t > Fe^{-r(1-t)}$, the investment in the stock remains unchanged; *ii*) If the portfolio value, at time t, is lower or equal to the floor present value, $P_t \leq Fe^{-r(1-t)}$, the stock is immediately sold out and the investor's wealth is invested into the risk-free asset.

The CPPI (Constant Proportion Portfolio Insurance) was originally proposed by Perold (1986) and Black and Jones (1987, 1988). The difference at each time t between the portfolio value (P_t) and the floor (F_t) is defined as the cushion. The product of the cushion for a multiple (m), gives us, at time t, the amount to apply in the risky asset, which is called the exposition. Over time, if the growth in the risky asset exceeds the risk-free rate of return, the cushion will rise and the investor's wealth should be switched from the risk-free to the risky asset, allowing the investor to retain exposure to higher returns. If the risky asset performance is not so good, the investor's wealth, in the rebalancing moments, will be transferred into the risk-free asset, allowing the investor to have a minimum value (floor) on his portfolio.

It is difficult to evaluate the different Portfolio Insurance strategies, because they are not utility maximizing and because of the widely spoken asymmetry of the expected returns. As in Garcia and Gould (1987), Bird et al. (1988, 1990) and Benninga (1990), we try to get our answers by empirical simulation against market data.

We choose to test the performance of Portfolio Insurance against actual market data, PSI-20 Index and DJ Stoxx 50 index. We have considered the data obtained between January 2003 and December 2008. We make cross-section comparisons between portfolios with the same starting value and which guarantee the same minimum value at the end of the period.

The Stop-Loss strategy is the one that presents better results when there is a rise in the indices. For the scenario where there is a decrease at the beginning of the year, followed by a gradual rise of the indices in the last months, OBPI provides the best results. CPPI seems to be more appropriate in scenarios where there is a big drop in the indices. The Stop-Loss and the CPPI strategies have the advantage of not using the option valuation theory, which makes these techniques less complex. But, we may not forget that there are scenarios where only OBPI allows obtaining the expected results.

We find that the technique performances are path-dependent and are not related to the method complexity degree. We also find that in some market conditions, the simplest techniques provide the best results.

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FUNDING OF A HYBRID PENSION SCHEME

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A Hybrid Pension Scheme, called the Modified Contribution (MC), is proposed for an individual accumulating a fund with variable contributions based on a pre-defined target. The aim of this work is to consider the effects of this MC fund on: 1) the value of deficits through time, 2) the fund and contribution variance and 3) the asset allocation through time which minimises the total future cost.

1. METHODOLOGY

The Modified Contribution model assumes an individual starting a pension scheme at age 25 (time 0) and retiring at age 65 (time 40). To simplify our model salary growth is not modelled. That is, the projected final salary at retirement is assumed to be equal to 1. The only source of unpredictable experience that is considered in this model is through volatile rates of return. Then, the MC pension fund is simulated by making use of the following recursion:

$$f_{t+1} = (f_t + C_t)(1 + i_{t+1}) \tag{1}$$

where i_{t+1} is the rate of investment return in year $(t, t+1)$ which is modelled considering two methods: the bootstrap sampling method with historical data for the period 1899 to 2001 and a dynamic asset allocation by assuming the alternative model described in Vigna and Haberman (2001).
The contribution in equation 1 that should be paid by the individual is as follows:

$$C_t = c + S_t \tag{2}$$

where c is a constant contribution calculated as a notional Defined Benefit (DB) scheme which provides a target benefit of 2/3 of final salary and S_t the adjustment to the contribution assumed to follow two methods which are compared in this work: the spreading described by Dufresne (1988), and the modified spreading (MS) developed by Owadally (2003)[1].

[1]The interested reader should refer to Gómez-Hernández (2008) for a full description of the model.

2. RESULTS

The main results of this work are as follows. First, that when an individual accumulates a Modified Contribution pension fund (MC), its value closely matches the value of the pre-defined target under the modified spreading method (MS). That is, the value of the deficits through time are closer to zero when assuming the MS than when assuming the spreading.
Second, the individual would prefer to adjust the value of his or her contributions by assuming the modified spreading method as the trade off between the volatility of the fund and the contribution can be decreased. That is, under the MS there is a choice on the value of the trade off between the variance of the fund and contribution. Then, depending on the individual's degree of risk aversion, an optimal choice of the variance can be made.
Third, a more conservative asset allocation through the working life of an individual than the so-called 'lifestyle' investment strategy, is found to give a smaller value of the total future cost within the fund of an individual. That is, regardless of the two methods assumed to adjust the value of the contributions, the percentage of the fund that should be invested in a high-risk asset, should be gradually increased till the end of the working life of the individual, in order to minimise the total future cost. This percentage is found to be smaller under the MS than under the spreading method.

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NON-PARAMETRIC ESTIMATION FOR MULTIVARIATE COMPOUND POISSON PROCESSES AND GOODNESS-OF-FIT TESTING

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We present a low-frequency estimator for the copula of the jump distribution of multivariate compound Poisson processes. Further we apply this estimator to construct a goodness-of-fit test for the jump copula.

1. MOTIVATION

In this paper, we are investigating multivariate compound Poisson processes (CPPs) with non-negative jumps, i.e. $X_t = \sum_{k=1}^{N_t} Y_k$, for all $t \geq 0$, where, as usual, $\{N_t\}_{t\geq 0}$ is a Poisson process with (known) intensity $\lambda > 0$ which is independent of the independent and identically distributed d-dimensional random jumps with distribution F. Based on low-frequency (compound) observations $X_1, \ldots, X_N$ we want to estimate the copula $C(x_1, \ldots, x_d) = F(F_1^{-1}(x_1), \ldots, F_d^{-1}(x_d))$ of these jumps. The multivariate estimator is a generalisation of a plug-in estimator for univariate CPPs constructed by Buchmann and Grübel (2003).
Recently Esmaeili and Klüppelberg (2008) estimated parametric Lévy copulas of CPPs including an application to insurance risk. To validate copulas several goodness-of-fit tests have been developed, yet no test for jump copulas of CPPs has been constructed. In the following section we present an estimator for the jump copula, while Section 3 is devoted to goodness-of-fit testing in this context. For detailed proofs, examples, and simulation results we refer to Schicks (2009).

2. NON-PARAMETRIC ESTIMATION OF THE JUMP COPULA

We start by denoting the empirical distribution function of the increments $\Delta X_j = X_j - X_{j-1}$ by

$$G_N(x) = \frac{1}{N} \sum_{j=1}^{N} \mathbb{I}_{[0,x]}(\Delta X_j) \qquad \text{for all } x \in \mathbb{R}_+^d.$$

Analogously we denote the empirical distribution functions of the margins by $G_{1,N}, \dots, G_{d,N}$. Further let $D_\tau = D_\tau([0,\infty]^d)$ denote the Banach space of d-dimensional càdlàg-functions with the norm $\|f\|_{\infty,\tau} = \sup_{x\in[0,\infty]^d} \left|e^{-\langle\tau,x\rangle} f(x)\right|$.
Let E_λ denote the compound distribution of the standard uniform distribution w.r.t. a Poisson process with intensity λ. Further, define the following decompounding operator

$$\Lambda_d(G) := \sum_{k=1}^{\infty} \frac{(-1)^{k+1} e^{\lambda k}}{\lambda k} (G^0)^{*k},$$

where $G^0(.) := G(.) - G(0)$. Then $\Lambda_d(G)$ converges in D_τ, if the Laplace transform of G^0 fulfils $\tilde{G}^0(\tau) < e^{-\lambda}$, which is always the case, if τ is chosen large enough. Now we can define an estimator C_N for the jump copula of a CPP, for $(x_1,\dots,x_d) \in [0,1]^d$, by

$$C_N(x_1,\dots,x_d) := \Lambda_d\big(G_N(G_{1,N}^{-1} \circ E_\lambda, \dots, G_{d,N}^{-1} \circ E_\lambda)\big)(x_1,\dots,x_d).$$

For $N \to \infty$ one can then show that the *empirical jump copula process* converges weakly to a centered Gaussian process in $l^\infty([p,q]^d)$, i.e.

$$\sqrt{N}\big(C_N(x_1,\dots,x_d) - C(x_1,\dots,x_d)\big) \rightsquigarrow Z(x_1,\dots,x_d),$$

where Z is a Gaussian process, whose covariance structure depends on C only. Moreover, $0 \le p,q \le 1$ need to be chosen in such a way, that C has continuous partial derivatives on $[p,q]^d$.

3. GOODNESS-OF-FIT TEST FOR THE JUMP COPULA

The asymptotic behaviour of the empirical jump copula process leads to the following result on Kolmogorov-Smirnovs tests for the copula C_0 of F, where C_0 is assumed to have partial derivatives on $[p,q]^d$. Define the following test statistic

$$T_{[p,q],N} = \sup_{(x_1,\dots,x_d)\in[p,q]^d} \sqrt{N}\big|C_N(x_1,\dots,x_d) - C_0(x_1,\dots,x_d)\big|.$$

Further let $z_{N,1-\alpha}$ denote the $(1-\alpha)$-quantile of T_N under C_0, then $P_C\big(T_N > z_{N,1-\alpha}\big) \to 1$, as $N \to \infty$, i.e. the Kolmogorov-Smirnov statistic is asymptotically pointwise consistent.

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De Koninklijke Vlaamse Academie van België voor Wetenschappen en Kunsten coördineert jaarlijks tot 25 wetenschappelijke bijeenkomsten, ook contactfora genoemd, in de domeinen van de natuurwetenschappen (inclusief de biomedische wetenschappen), menswetenschappen en kunsten. De contactfora hebben tot doel Vlaamse wetenschappers of kunstenaars te verenigen rond specifieke thema's.

De handelingen van deze contactfora vormen een aparte publicatiereeks van de Academie.

De traditie van de "Actuarial and Financial Mathematics" contactfora werd ook dit jaar verder gezet met de "Actuarial and Financial Mathematics Conference 2009" (kortweg AFMathConf2009) waarbij opnieuw aandacht werd besteed aan de interactie tussen financieel en actuarieel wiskundige technieken. Naast genodigde sprekers en de bijdragen hadden de organisatoren geopteerd voor twee short courses over *Solvency II* en over *Lévy processes* en een postersessie. In deze postersessie kregen heel wat jonge onderzoekers de mogelijkheid om hun onderzoeksresultaten voor te stellen aan een ruim publiek bestaande uit academici uit binnen- en buitenland alsook collega's uit de bank- en verzekeringswereld. In deze publicatie vindt u de cursusnota's van de short course over *Lévy processes.* Verder twee bijdragen over het prijzen van "death bonds" en van FX, inflatie en aandelenopties onder stochastische rentevoeten en stochastische volatiliteit. Tenslotte bevat deze publicatie de uitgebreide abstracts van de meeste postervoorstellingen.