KONINKLIJKE VLAAMSE ACADEMIE VAN BELGIE
VOOR WETENSCHAPPEN EN KUNSTEN

ACTUARIAL AND FINANCIAL MATHEMATICS CONFERENCE

Interplay between Finance and Insurance

February 10-11, 2011

Michèle Vanmaele, Griselda Deelstra, Ann De Schepper, Jan Dhaene, Steven Vanduffel & David Vyncke (Eds.)

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Actuarial and Financial Mathematics Conference
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CONTENTS

Invited talk

Contributed talks

Poster session

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Actuarial and Financial Mathematics Conference
Interplay between finance and insurance

PREFACE

The yearly meeting “Actuarial and Financial Mathematics Conference” of academic researchers and practitioners to discuss recent developments at the interplay between finance and insurance took place on February 10 and 11, 2011 as a contactforum in the buildings of the Royal Flemish Academy of Belgium for Science and Arts in Brussels. At this occasion researchers in actuarial and financial mathematics from Belgian universities and from abroad on the one side, and professionals of the banking and insurance business on the other side took the opportunity to get in touch with each other or to strengthen the ties. The number of conference partcipants is each year growing. For the AFMathConf2011 about 130 participants from 21 different countries were registered, illustrating the large interest in this event.

For this 2011 edition, eight internationally esteemed researchers were invited to give an overview talk on their recent research topic. During the first day, we welcomed *Damir Filipovic (Ecole Polytechnique Fédérale de Lausanne and Swiss Finance Institute, Switzerland), Carole Bernard (University of Waterloo, Canada), Rama Cont (CNRS, France & Columbia University, USA) and Andreas Kyprianou (University of Bath, UK).* Their very clear expositions gave the audience some insight in a quadratic term structure model for the variance swap rates, in the explicit construction of an optimal strategy when investors have state-dependent constraints, in an approach to measuring systemic risk based on explicit modeling of counterparty relations between financial institutions as a weighted graph and in some of the remarkable developments that have occurred in the development of ruin theory and de Finetti’s classical control problem in the last two or three years. The second day, the attendants had the opportunity to listen to the following invited speakers: *Mario Wüthrich (ETH Zurich, Switzerland), Pierre Devolder (Université catholique de Louvain, Belgium), Alexander McNeil (Heriot-Watt University, Scotland) and Giulia Di Nunno (University of Oslo, Norway).* The first three talks dealt with insurance issues such as a novel stochastic model for claims reserving that allows to combine claims payments and incurred losses information, some various stochastic models in continuous time in order to estimate solvency capital for two important risks faced by pension funds: market risk and inflation risk and an approach to multivariate stress testing for solvency. Optimal investment in assets subject to risk of default for investors that rely on different levels of information was the topic of the last invited talk.

Next to the invited lectures, the scientific committee selected eight contributions which were spread over the two days. These talks with topics in finance and insurance were given by *Florence Guillaume (T.U.Eindhoven, The Netherlands), Łukasz Delong (Warsaw School of*

Economics, Poland), Eva Lütkebohmert (University of Freiburg, Germany), Kathrin Glau (University of Vienna, Austria), Robert Salzmann (ETH Zurich, Switzerland), Elisa Luciano (Università degli Studi di Torino, Italy), Catherine Donnelly (Heriot-Watt University, Scotland) and Zorana Grbac, University of Evry, France). In addition, twelve researchers presented a poster during an appreciated poster session. We thank them all for their enthusiasm and their nice presentations which made the conference a great success.

The present proceedings give an overview of the activities at the conference. They contain one article related to an invited talk, five papers corresponding to contributed talks, and ten short communications of posters presented during the poster sessions on both conference days.

We are much indebted to the members of the scientific committee, *Hansjörg Albrecher (HEC Lausanne, Switzerland), Freddy Delbaen (ETH Zurich, Switzerland), Michel Denuit (Université Catholique de Louvain, Belgium), Ernst Eberlein (University of Freiburg, Germany), Rob Kaas (University of Amsterdam, the Netherlands), Ragnar Norberg (London School of Economics, UK), Noel Veraverbeke (Universiteit Hasselt, Belgium) and the chair Griselda Deelstra (Université Libre de Bruxelles, Belgium),* for the excellent scientific support. We also thank *Wouter Dewolf (Ghent University, Belgium),* for the administrative work.

We cannot forget our sponsors, who made it possible to organise this event in a very enjoyable and inspiring environment. We are very grateful to the Royal Flemish Academy of Belgium for Science and Arts, the Research Foundation ─ Flanders (FWO), the Scientific Research Network (WOG) “Fundamental Methods and Techniques in Mathematics”, le Fonds de la Recherche Scientifique (FNRS), Dexia, Electrabel-GDF Suez and the BNP Paribas Fortis Chair in Banking at The Vrije Universiteit Brussel and Université Libre de Bruxelles.

The success of the meeting encourages us to go on with the organisation of this contactforum. We are sure that continuing this event will provide more opportunities to facilitate the exchange of ideas and results in our fascinating research field.

The editors:
Griselda Deelstra
Ann De Schepper
Jan Dhaene
Steven Vanduffel
Michèle Vanmaele
David Vyncke

The other members of the organising committee:
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INVITED TALK

OPTIMAL INVESTMENT UNDER PROBABILITY CONSTRAINTS

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Abstract

Bernard and Boyle (2010) derive the lowest cost strategy (also called "cost-efficient" strategy) that achieves a given wealth distribution. An optimal strategy for a profit seeking investor with law-invariant preferences is necessarily cost-efficient. In the specific case of a Black-Scholes market the optimal strategy is always path-independent and non-decreasing with the stock price. Assuming now that investors still want to achieve a given distribution at a fixed horizon but have a probability constraint, we propose an explicit construction of the optimal strategy. In the case of the Black-Scholes market, we show that the optimal strategy is not necessarily non-decreasing in the stock price any more.

1. INTRODUCTION

This note extends Bernard and Boyle (2010) by including additional probability constraints. An investor with law-invariant preferences but with some probability constraints has "state-dependent" preferences. We show that the non-decreasing property of the optimal investment for law-invariant preferences does not hold when preferences are state-dependent. Section 2 gives our assumptions, the framework and recalls what cost-efficiency is and its link with optimal investment. Section 3 provides some theoretical results on bounds on copulas under probability constraints and how to use them to solve our optimization problem. We apply theoretical results of Section 3 to some optimal investment problems in Section 4.

[1]Both authors gratefully acknowledge the program "Brains Back to Brussels" that funded an extended research visit of C. Bernard at VUB in Brussels during which this paper was completed. S. Vanduffel acknowledges the financial support of the BNP Paribas Fortis Chair in Banking. C. Bernard also acknowledges support from WatRISQ and the Natural Sciences and Engineering Research Council of Canada.

2. COST-EFFICIENCY & OPTIMAL INVESTMENT

In this section we first present the model assumptions and the setting. We then give the general form of the optimal investment problem we want to solve in this paper. In particular we relate the optimal investment choice to the concept of "cost-efficiency" (originally defined by Dybvig (1988a,b)).

2.1. Agent's Preferences

Denote by $U(\cdot)$ the investor's objective function he wants to maximize. We make the following assumptions.

- All investors have a fixed investment horizon $T > 0$ and there is no intermediate consumption.
- Investors prefer "more to less", in other words their respective objective functions preserve first order stochastic dominance relationships (denoted by $\prec_{fsd}$). Hence if $Y_T \prec_{fsd} X_T$ then $U(X_T) \geqslant U(Y_T)$ and $U(\cdot)$ is non-decreasing.
- Investors have "state-independent preferences" or "law-invariant preferences": if Y_T has the same distribution as X_T then $U(Y_T) = U(X_T)$.

Such set of preferences is quite general and consistent with a wide range of decision theories, including the expected utility theory (von Neumann and Morgenstern (1947)), Yaari's dual theory of choice (Yaari (1987)), the cumulative prospect theory (Tversky and Kahneman (1992)) and the rank dependent utility theory (Quiggin (1993)). For example, in the particular case of expected utility the preferences for a final wealth X_T would be calculated as $U(X_T) = \mathbb{E}[u(X_T)]$ where u is the investor's utility function. Instead of maximizing an objective function, one may also minimize any law-invariant risk measure that preserves first stochastic dominance (for example the quantile or a general distorted expectation).

2.2. Financial Market

The financial market contains a (risk-free) bond with price process $\{B_t = B_0 e^{rt},\ t \geqslant 0\}$. Further, there is also a risky asset S with price process $\{S_t,\ t \geqslant 0\}$. We assume trading can be done continuously, the market is frictionless and arbitrage-free, and all investors agree on the pricing kernel used to value derivatives in this market. The initial price $c(X_T)$ of a given contract with payoff X_T maturing at the fixed horizon $T > 0$ is given by

$$c(X_T) = \mathbb{E}[\xi_T X_T]. \tag{1}$$

Here the expectations are taken with respect to the physical probability measure $\mathbb{P}$, and $\{\xi_t,\ t \geqslant 0\}$ is called the state-price process. We will also assume that ξ_t is continuously distributed. In particular it holds that

$$c(1) = \mathbb{E}[\xi_T] = e^{-rT}. \tag{2}$$

It is also well-known that $c(X_T)$ can be presented as the discounted expectation under the risk-neutral measure $\mathbb{Q}$ defined through $\xi_t = e^{-rt}(\frac{d\mathbb{Q}}{d\mathbb{P}})_t$. In the remainder of the paper all expectations are taken under the $\mathbb{P}$ measure. We refer to Bjork (2004) for extensive theory on arbitrage-free pricing.

Note that the above description is rather general and includes the Black-Scholes setting in which case the process $\{\xi_T, t \geqslant 0\}$ is known unambiguously. For the ease of exposition we present all the results in the one-dimensional Black-Scholes market[2]. In this setting there is a bijection between the state-price process ξ_t and the risky asset S_t. Recall that the risky asset price S_t evolves according to

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t, \tag{3}$$

where $\{W_t,\ t \geqslant 0\}$ is a standard $\mathbb{P}$-Brownian motion and assume $\mu > r$. The state price process $\{\xi_t,\ t \geqslant 0\}$ exists, is unique and is given by

$$\xi_t = a\left(\frac{S_t}{S_0}\right)^{-\frac{\theta}{\sigma}}, \tag{4}$$

where $a = e^{\frac{\theta}{\sigma}(\mu - \frac{\sigma^2}{2})t - (r + \frac{\theta^2}{2})t}$ and $\theta = \frac{\mu - r}{\sigma}$. Note that ξ_t is decreasing in S_t. Denote by F_ξ the cdf of ξ_T. Let M_T denote the mean of $\log(\xi_T)$, $M_T = -\frac{1}{2}\theta^2 T - rT$. The variance of $\log(\xi_T)$ is equal to $\theta^2 T$. Then,

$$F_\xi(x) = P\left(\xi_T \leq x\right) = \Phi\left(\frac{\log(x) - M}{\theta\sqrt{T}}\right). \tag{5}$$

2.3. Cost Efficiency & Investment

The concept of "cost-efficiency" was first introduced by Cox and Leland (1982, 2000) and Dybvig (1988a,b).

Definition 2.1 *A strategy (or a payoff) is cost-efficient if any other strategy that generates the same distribution costs at least as much.*

It is clear that if investors prefer more to less (as per our assumptions in Section 2.1), then in the absence of additional constraints optimal investment strategies will necessarily be cost-efficient. Given the cdf that the investor would like to achieve at a given maturity date T (possibly a retirement date), the optimal strategy then solves the following problem

$$(\mathcal{P}_1) \qquad \begin{array}{l} \min\limits_{X_T} \ \mathbb{E}\left[\xi_T X_T\right] \\ \text{subject to } \ \forall x \in \mathbb{R},\ \ \mathbb{P}(X_T \leqslant x) = F(x) \end{array} \tag{6}$$

The objective is to minimize the cost of a payoff X_T such that X_T has cdf F. Define F^{-1} as follows

$$F^{-1}(y) = \inf\left\{x \mid F(x) \geqslant y\right\}.$$

[2] It would be possible to be more general and include the multidimensional case as studied by Bernard et al. (2011) or the Levy market presented in Vanduffel et al. (2011).

The inverse is left-continuous and non-decreasing. Theorem 2.1 characterizes the optimal investment strategy.

Theorem 2.1 *Let F be a cdf. The solution to $\mathcal{P}_1$ given by* (6) *is equal to*

$$Y_T^\star = F^{-1}\left(1 - F_\xi\left(\xi_T\right)\right), \tag{7}$$

and it is the almost surely unique optimal solution to (6).

This theorem corresponds to the main result of Bernard and Boyle (2010). We will see that it can be obtained as a special case of our approach.

Assume now that the investor is subject to additional constraints that are "state-dependent". The cost-efficient strategy (7) solution to $\mathcal{P}_1$ may not satisfy these constraints and therefore the optimal strategy may be strictly more expensive. We formulate the problem as follows.

$$(\mathcal{P}_2) \qquad \begin{array}{l} \min\limits_{X_T} \ \mathbb{E}\left[\xi_T X_T\right] \\ \text{subject to} \ \begin{cases} \forall x \in \mathbb{R}, \ \ \mathbb{P}(X_T \leqslant x) = F(x) \\ (\mathcal{C}_i)_{i\in I} \end{cases} \end{array} \tag{8}$$

The optimal strategy is distributed with the cdf F but in addition each $\mathcal{C}_i$ denotes an additional constraint and I can be finite or infinite. Each constraint $\mathcal{C}_i$ contains information about the dependency structure between the state-price process and the optimal strategy of the investor given by

$$\mathbb{P}(\xi_T < \ell_i, X_T < x_i) = b_i.$$

In a Black-Scholes market, the state-price process is a function of the risky asset. Then a natural example is a simple probability constraint ensuring that the investment strategy is greater than some guaranteed level when the market itself is very low. The constraint can then write as

$$\mathbb{P}(S_T < \alpha S_0, X_T > b) \leqslant \varepsilon,$$

where $\alpha < 1$, see equation (4).

Adding such constraints is important because investors have state-dependent constraints. For example an investor who invests in a put option, is not interested in cost-efficiency only (because it is decreasing in the underlying stock) but wants positive outcomes when the market goes down.

3. SOLUTIONS TO PROBLEMS $(\mathcal{P}_1)$ AND $(\mathcal{P}_2)$

3.1. Formalization

Problems $\mathcal{P}_1$ and $\mathcal{P}_2$ presented above can be reformulated as "dependence" problems (in other words as problems on copulas). Indeed Problem $\mathcal{P}_1$ is clearly a minimization of $\mathbb{E}[X_T\xi_T]$ where marginals of X_T and ξ_T are known but where no information about the dependency between X_T and ξ_T is given. It can also be interpreted as the minimization of $\mathbb{E}[X_T g(S_T)]$ where marginals of S_T and X_T are known and where $g(y) = a(y/S_0)^{-b}$ for some $b > 0$ because of (4). Problem $\mathcal{P}_2$ is similarly a minimization of $\mathbb{E}[X_T\xi_T]$ or $\mathbb{E}[X_T g(S_T)]$ but with some information on the dependency between X_T and the market S_T.

Let (X, Y) be a couple of random variables. It is well-known that the joint distribution for (X, Y) is fully determined upon knowledge of the marginal distributions F_X and F_Y together with the copula function $C := C_{(X,Y)}$ for (X, Y) (this result is known as Sklar's theorem).

Let us define supermodular functions. Let $\underline{e}_i$ denote the i-th $n$dimensional unit vector, and let $f : \mathbb{R}^n \to \mathbb{R}$ be some function. For $\underline{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$ we then define $\Delta_i^\varepsilon f(x) = f(\underline{x} + \varepsilon\, \underline{e}_i) - f(\underline{x})$ ($\varepsilon_i > 0, 1 \leqslant i \leqslant n$).

Definition 3.1 (Super modularity) *A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be supermodular (or 2-increasing) if for all $\underline{x} \in \mathbb{R}^n$, $\delta > 0$, $\varepsilon > 0$ and $1 \leqslant i < j \leqslant n$ it holds that.*

$$\Delta_i^\delta \Delta_j^\varepsilon\, f(\underline{x}) \geqslant 0.$$

If $f : \mathbb{R}^n \to \mathbb{R}$ is twice differentiable then f is supermodular if and only if $\frac{\partial^2}{\partial x_i \partial x_j} f\,(\underline{x}) \geqslant 0$ holds for every $\underline{x} \in \mathbb{R}^n$ and $1 \leqslant i < j \leqslant n$.

See for example Marshall and Olkin (1979), p. 146. A function f is submodular when $-f$ is supermodular.

The problem $\mathcal{P}_2$ given in (8) we want to solve amounts to studying integrals of the form $\mathbb{E}[f(X,Y)]$ where f is submodular or supermodular. Theorem 3.1 below can be found in Tankov (2011) and provides, under suitable assumptions, an expression for the integral $\mathbb{E}[f(X,Y)]$ in terms of the copula C, and the marginal distributions F_X and F_Y.

Theorem 3.1 (Bounds for $\mathbb{E}[f(X,Y)]$) *Assume $f : \mathbb{R}_+^2 \to \mathbb{R}$ is supermodular and left-continuous in each of the arguments. Assume also that*

$$\mathbb{E}\left[|f(X,0)| + |f(0,X)| + |f(Y,0)| + |f(0,Y)| + |f(X,X)| + |f(Y,Y)|\right] < \infty,$$

then $\Pi(C) = \mathbb{E}[f(X,Y)]$ is given by

$$\begin{aligned} \Pi(C) &= -f(0,0) + \mathbb{E}\left[f(X,0)\right] + \mathbb{E}\left[f(0,Y)\right] \\ &\quad + \int_0^\infty \int_0^\infty \mu_f(dx \times dy)(1 - F_X(x) - F_Y(y) + C(F_X(x), F_Y(y)) \end{aligned} \tag{9}$$

where μ_f is the measure on $\mathbb{R}_+^2$ induced by the supermodular function f.

In addition, if the copula C admits pointwise bounds L and U

$$\forall u \in (0,1), \forall v \in (0,1) \quad L(u,v) \leqslant C(u,v) \leqslant U(u,v).$$

Then

$$\Pi(L) \leqslant \Pi(C) \leqslant \Pi(U), \tag{10}$$

where L and U are not necessarily copulas but could be more general functions (such that the double integral in (9) *exists).*

Proof. The expression (9) is given in Proposition 2 of Tankov (2011). ■

It seems that the expression (9) did not appear yet elsewhere in the literature although it is not the focus of Tankov (2011)[3]. As a first application of Theorem 3.1 let us consider the supermodular function f defined as $f(x,y) = xy$. In this case $\mu_f(dx \times dy) = dx \times dy$. Hence

$$\mathbb{E}[XY] = \int_0^\infty \int_0^\infty \mathbb{P}(X > x, Y > y)dxdy, \tag{11}$$

which is well-known.

Another example of supermodular function is $f(x,y) = -xg(y)$ where $g(y) = a \cdot (y/S_0)^{-b}$. This function appears in the case of a one dimensional Black-Scholes market as the bijection between the risky asset (respectively the market portfolio) and the state price process. In this case, the objective to minimize in problems $\mathcal{P}_1$ and $\mathcal{P}_2$ corresponds to minimizing $\mathbb{E}[f(X_T, S_T)]$. Note that $\frac{\partial^2 f}{\partial x \partial y} \leqslant 0$ which means that it is a submodular function. In that case, $\mu_f(dx \times dy) = g'(y)dxdy$. Hence

$$\mathbb{E}[Xg(Y)] = \int_0^\infty \int_0^\infty \mathbb{P}(X > x, Y > y)g'(y)dxdy. \tag{12}$$

Theorem 3.1 is very useful to actually compute bounds for $\mathbb{E}[f(X,Y)]$ in case one knows the marginal distributions of X and Y, with limited information on the dependence between X and Y. The main idea is to translate the information one has on the dependence to derive bounds on the unknown copula $C_{(X,Y)}$. Using Theorem 3.1 (precisely the inequality (10)), solving problems $\mathcal{P}_1$ and $\mathcal{P}_2$ amounts to finding bounds on copulas. Problem $(\mathcal{P}_1)$ given in (6) and Problem $(\mathcal{P}_2)$ given in (8) can then be formulated as special cases of the following general problem

$$\begin{aligned} &\min_X \ \mathbb{E}\left[f(X,Y)\right] \\ &\text{subject to } \begin{cases} X \sim F, Y \sim G \\ \forall i \in I, \quad \mathbb{P}(Y < \ell_i, X < x_i) = b_i \end{cases} \end{aligned} \tag{13}$$

where I is the set of constraints. Problem $\mathcal{P}_1$ corresponds to $I = \emptyset$. Each additional constraint directly provides information on the dependence between X and Y. In Problem $\mathcal{P}_1$ and $\mathcal{P}_2$, the r.v. Y is the state-price process or a function of S_T, its distribution G is known and depends on the financial market.

The rest of the paper focuses on deriving the bounds A and B such that the unknown copula between X and Y satisfies

$$\forall u, v \in (0,1), \quad A(u,v) \leqslant C_{(X,Y)}(u,v) \leqslant B(u,v) \tag{14}$$

---

[3]It generalizes many existing formulas in the literature. For example consider the supermodular function f, $f(x,y) = (x+y-d)_+$. In this case we obtain: $\mu_f(dx \times dy) = \delta_{y=d-x}.dx \times dy$. Hence

$$\begin{aligned} \mathbb{E}[(X+Y-d)_+] &= \mathbb{E}\left[(X-d)_+\right] + \mathbb{E}\left[(Y-d)_+\right] + \int_0^d \mathbb{P}(X > x, Y > d-x)dx \\ &= \mathbb{E}[X] + \mathbb{E}[Y] - d + \int_0^d \mathbb{P}(X \leqslant x, Y \leqslant d-x))dx \end{aligned}$$

which conforms with the expression for $\mathbb{E}[(X+Y-d)_+]$ that was derived in Dhaene and Goovaerts (1996). Their result now appears as a special case of Theorem 3.1.

In general the bounds A and B are not copulas but quasi-copulas. First recall that a two-dimensional copula is any supermodular function $C : [0,1]^2 \to [0,1]$ such that for all u $\in (0,1)$ it holds that $C(0,u) = C(u,0) = 0$ and also that $C(u,1) = C(1,u)$. It is well-known that this definition implies that C is increasing in each argument and also that C is Lipschitz continuous, i.e that $|C(u_1,v_1) - C(u_2,v_2)| \leqslant |u_1 - u_2| + |v_1 - v_2|$ for all (u_1,v_1), $(u_2,v_2) \in [0,1]^2$. These two properties together with the boundary conditions define the weaker concept of quasi-copula:

Definition 3.2 (Quasi-copula) *A two-dimensional quasi-copula is any function $Q : [0,1]^2 \to [0,1]$ with the following properties:*

(i) Boundary conditions: for all u $\in (0,1)$ it holds that $Q(0,u) = Q(u,0) = 0$ and also that $Q(u,1) = Q(1,u)$;

(ii) Q is increasing in each argument and Lipschitz continuous.

Of course any copula is a quasi-copula but the opposite is not true; for an insightful treatment of copulas we refer to Nelsen (2006). For example a characterization of quasi-copulas is given in Theorem 2.1 of Nelsen et al. (2002).

3.2. Solution to $\mathcal{P}_1$

In Problem $\mathcal{P}_1$, the marginal distributions F_X and F_Y are known but no information is given.

Theorem 3.2 (Classical Fréchet bounds) *Consider a random couple (X,Y), it is well-known that*

$$\forall u, v \in (0,1), \quad \min(u,v) \leqslant C(u,v) \leqslant \max(0, u+v-1)$$

which respectively correspond to the comonotonic and anti-comonotonic copula. Let f be a supermodular function. Then,

$$\mathbb{E}\left[f\left(F_X^{-1}(U),\, F_Y^{-1}(1-U)\right)\right] \leqslant \mathbb{E}[f(X,Y)] \leqslant \mathbb{E}\left[f\left(F_X^{-1}(U),\, F_Y^{-1}(U)\right)\right].$$

Proof. This result is well-known and the proof is omitted. ■

Solving Problem $\mathcal{P}_1$ is now straightforward and Theorem 2.1 can be seen as a particular case of Theorem 3.2 where $f(x,y) = xy$. For every X_T with cdf F it holds that

$$E[F^{-1}(1 - F_{\xi_T}(\xi_T))] \leqslant \mathbb{E}[\xi_T X_T] \leqslant \mathbb{E}[F^{-1}(F_{\xi_T}(\xi_T))] \tag{15}$$

Note that $(U, 1-U)$ is a legitimate copula so that the bounds are reached.

3.3. Solution to $\mathcal{P}_2$ under probability constraints

We assume that the information on the dependence between X and Y is such that the copula $C_{(X,Y)}$ is known on a compact subset of the unit square. Bounds were given by Tankov (2011) and we recall here his results

Theorem 3.3 *Let $\mathcal{S}$ be a compact subset of $[0,1]^2$ and consider a quasi-copula Q. Let us define for all u,v $\in [0,1]$*

$$\begin{aligned} U^{\mathcal{S},Q}(u,v) &= \min\left(u, v, \min_{(a,b)\in\mathcal{S}} \left\{Q(a,b) + (u-a)^+ + (v-b)^+\right\}\right), \\ L^{\mathcal{S},Q}(u,v) &= \max\left(0, u+v-1, \max_{(a,b)\in\mathcal{S}} \left\{Q(a,b) - (a-u)^+ - (b-v)^+\right\}\right) \end{aligned} \tag{16}$$

Then for every quasi-copula Q_* so that $Q_*(a,b) = Q(a,b)$ for all $(a,b) \in \mathcal{S}$ it holds that for all $u, v \in [0,1]$

$$L^{\mathcal{S},Q}(u,v) \leqslant Q_*(u,v) \leqslant U^{\mathcal{S},Q}(u,v). \tag{17}$$

Furthermore for all $(a,b) \in \mathcal{S}$ we have that

$$L^{\mathcal{S},Q}(a,b) = U^{\mathcal{S},Q}(a,b) = Q(a,b). \tag{18}$$

Moreover $L^{\mathcal{S},Q}$ and $U^{\mathcal{S},Q}$ are quasi-copulas. Finally, when $\mathcal{S}$ is increasing and Q is a copula, we have that $L^{\mathcal{S},Q}$ is a copula whereas if $\mathcal{S}$ is decreasing, we have that $U^{\mathcal{S},Q}$ is a copula.

Proof. The proof can be found in Tankov (2011). ∎

Note that Theorem 3.3 can be applied whenever the values of a copula C are known on a compact subset $\mathcal{S}$ (C just plays the role of Q in this case).

Special case where $\mathcal{S}$ =$\{a,b\}$.
Let C_* a copula such that $C_*(a,b) = \vartheta$ with ϑ such that $\max(a+b-1,0) \leqslant \vartheta \leqslant \min(a,b)$ holds. Then for all $u, v \in [0,1]$ the upper and lower bounds are now given by

$$\begin{aligned} U^{a,b,\vartheta}(u,v) &= \min\left(u, v, \vartheta + (u-a)^+ + (v-b)^+\right), \\ L^{a,b,\vartheta}(u,v) &= \max\left(0, u+v-1, \vartheta - (a-u)^+ - (b-v)^+\right) \end{aligned} \tag{19}$$

respectively. Both are copulas and satisfy $L^{a,b,\vartheta}(a,b) = U^{a,b,\vartheta}(a,b) = C_*(a,b) = \vartheta$. These copulas are called shuffles. In short, a shuffle copula has a support constituted of line segments of slope +1 and -1. More details on shuffles are presented in Section 3.2.3 of Nelsen (2006).

4. EXAMPLES IN BLACK SCHOLES

4.1. Optimization with a unique probability constraint $C(a,b) = \vartheta$

We now describe the simulation of a couple of uniform random variables (U,V) with copula equal to the lower or upper bound found in (19). Draw first a random number u from the uniform (0,1) distribution, then V is fully determined. To obtain a couple (U,V) with the copula $L^{a,b,\vartheta}$, v is calculated as the following function of u

$$\begin{cases} v = 1-u & \text{if } 0 \leqslant u \leqslant a-\vartheta, \\ v = a+b-\vartheta-u & \text{if } a-\vartheta \leqslant u \leqslant a, \\ v = 1+\vartheta-u & \text{if } a \leqslant u \leqslant 1+\vartheta-b, \\ v = 1-u & \text{if } 1+\vartheta-b \leqslant u \leqslant 1. \end{cases} \tag{20}$$

For $U^{a,b,\vartheta}$, it is similar and omitted here. Panel A of Figure 1 gives the support of the shuffle copula $L^{a,b,\vartheta}$.

We now apply this to the construction of the "optimal" solution to $\mathcal{P}_2$ when the probability constraint is given by

$$\mathbb{P}(S_T < \alpha S_0, X_T > b) = \varepsilon \tag{21}$$

where $\alpha > 0$. This probability constraint ensures that the realized payoff is greater than some guaranteed level b when the market itself is low (case when $\alpha < 1$).

In the Black-Scholes model, $S_T = g(\xi_T)$ where g is non-increasing therefore

$$\begin{aligned}\mathbb{P}(S_T < \alpha S_0, X_T > b) &= \mathbb{P}(\xi_T > \ell, X_T > b)\\ &= \mathbb{P}(G(\xi_T) > G(\ell); F(X_T) > F(b))\\ &= 1 - G(\ell) - F(b) + C(G(\ell), F(b))\end{aligned}$$

where $\ell = g(S_0)$ and where C is the copula of (ξ_T, X_T). We are solving a special case of the problem $(\mathcal{P}_2)$ given in (8),

$$\min_{X_T} \ \mathbb{E}\left[\xi_T X_T\right]$$
$$\text{subject to} \ \begin{cases} X_T \sim F \\ \ln(S_T) \sim \mathcal{N}\left(\ln(S_0) + \left(\mu - \frac{\sigma^2}{2}\right) T, \sigma^2 T\right) \\ \mathbb{P}(S_T < \alpha S_0, X > b) = \varepsilon \end{cases}$$

This can be rewritten in terms of the state-price process. Note then that $\mathbb{P}(\xi_T \leqslant \ell, X_T \leqslant b) = \varepsilon - 1 + F_{\xi_T}(\ell) + F(b)$. Therefore the problem can be restated as

$$\min_{X_T} \ \mathbb{E}\left[\xi_T X_T\right]$$
$$\text{subject to} \ \begin{cases} X_T \sim F \\ \ln(\xi_T) \sim \mathcal{N}\left(M_T, V_T\right) \\ C(F_{\xi_T}(L), F(y_0)) = \vartheta \end{cases}$$

where $\vartheta = \varepsilon - 1 + F_{\xi_T}(\ell) + F(b)$ and C is the copula between ξ_T and X_T. We will use Theorem 3.1 where the copula C that appears in the formula is replaced by the copula L of the lower bound. We construct explicitly the optimal strategy by simulating $U = F_{\xi_T}(\xi_T)$ and constructing V following (20) to simulate a couple (U, V) of uniform (0,1) such that the copula is $L^{F_{\xi_T}(\ell), F(b), \vartheta}$. V is a function of U, let h be such that $V = h(U)$. Then the optimal solution to $\mathcal{P}_2$ with the probability constraint (21) given explicitly by

$$F^{-1}(h(F_\xi(\xi_T))).$$

4.2. Example when F is the cdf of a put option and there is one constraint.

Consider a put option with strike K and maturity T, its payoff is $X_T = (K - S_T)^+$. The cost efficient strategy was found in Bernard and Boyle (2010). We first recall their result and study the effect of adding the probability constraint. Let F be the cdf of the payoff of the put option.

Bernard and Boyle (2010) show that the put option is the (a.s.) unique payoff that has the highest possible cost with cdf F. This cdf, F, is

$$F(x) = P(X_T \leqslant x) = \begin{cases} 1 & \text{if } x \geqslant K \\ P(S_T > K - x) = \Phi\left(\frac{\left(\mu - \frac{\sigma^2}{2}\right)T - \log\left(\frac{K-x}{S_0}\right)}{\sigma\sqrt{T}}\right) & \text{if } 0 \leqslant x < K \\ 0 & \text{if } x < 0 \end{cases}$$

It is straightforward to invert it. Define $\nu = \Phi\left(\frac{\left(\mu - \frac{\sigma^2}{2}\right)T - \log\left(\frac{K}{S_0}\right)}{\sigma\sqrt{T}}\right)$ and consider $y \in (0, 1)$,

$$F^{-1}(y) = \left(K - S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)T - \sigma\sqrt{T}\Phi^{-1}(y)}\right)^+$$

Note that $F^{-1}(1) = K$ and $F^{-1}(0)$ is not well defined. The cost-efficient payoff that gives the same distribution as a put option is

$$Y_T^\star = F^{-1}\left(1 - F_\xi\left(\xi_T\right)\right) = \left(K - S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)T - \sigma\sqrt{T}\left(\frac{M - \log(\xi_T)}{\theta\sqrt{T}}\right)}\right)^+ = \frac{K}{S_T}\left(S_T - \frac{c}{K}\right)^+,$$

where F_ξ is given by (5) (see Theorem 2.1) and where $c = S_0^2 e^{2\left(\mu - \frac{\sigma^2}{2}\right)T}$. $Y_T^\star$ is the optimal solution to $(\mathcal{P}_1)$ (cheapest strategy with cdf F). We now want the cheapest strategy X_T with cdf F and

$$\mathbb{P}\left(X_T > b\,;\ S_T < 0.95 S_0\right) = \varepsilon$$

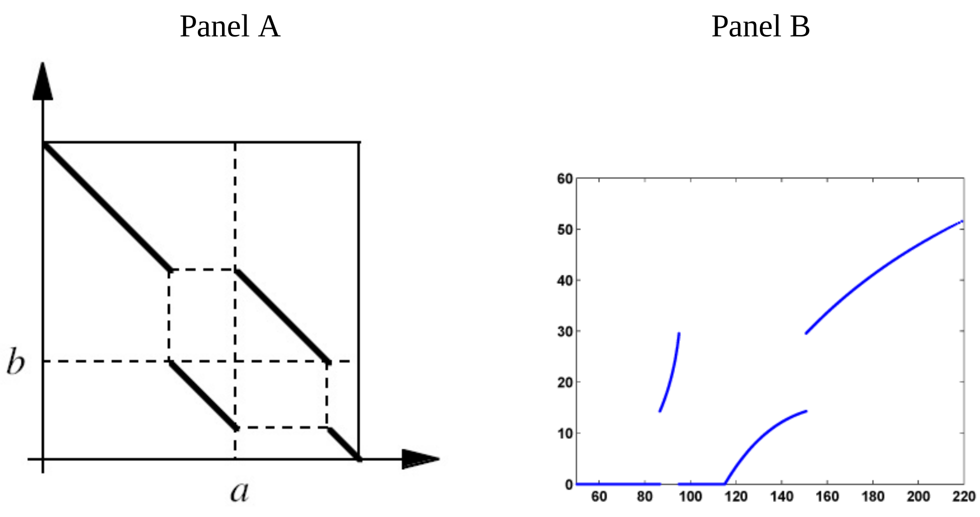

Figure 1: Panel A corresponds to the support of the copula $L^{a,b,\vartheta}$ given by (19). This is an extract from Fig. 3.10 in Nelsen (2006). Panel B displays the cheapest strategy as a function of S_T under the probability constraint under study. Assumptions for Panel B are: $S_0 = 100$, $K = 100$, $\mu = 0.05$, $\sigma = 0.2$, $T = 1$, $r = 0.03$, $b = K/7$ and $\varepsilon = .15$.

Panel B in Figure 1 illustrates the optimum.

4.3. Example when F is the cdf of a put option and there is an infinite number of constraints

With several probability constraints, we can solve $(\mathcal{P}_2)$ using the general result in Theorem 3.3. Assume that for all $(a, b) \in I$, the copula between ξ_T and X_T is comonotonic and therefore the copula between X_T and S_T is anti-comonotonic.

$$C(a, b) = \min(a, b)$$

where I is the segment with extremities $(0.7, 0.7)$ and $(1, 1)$. The constraint on the copula applies for $S_T \leq 92.8$ and $X_T \geq 7.21 = F^{-1}(0.7)$. We are looking for the cheapest strategy X_T with cdf F and X_T is anti-comonotonic with the stock market when the stock price is low.

The following figure gives the support of the copula L and the optimal strategy.

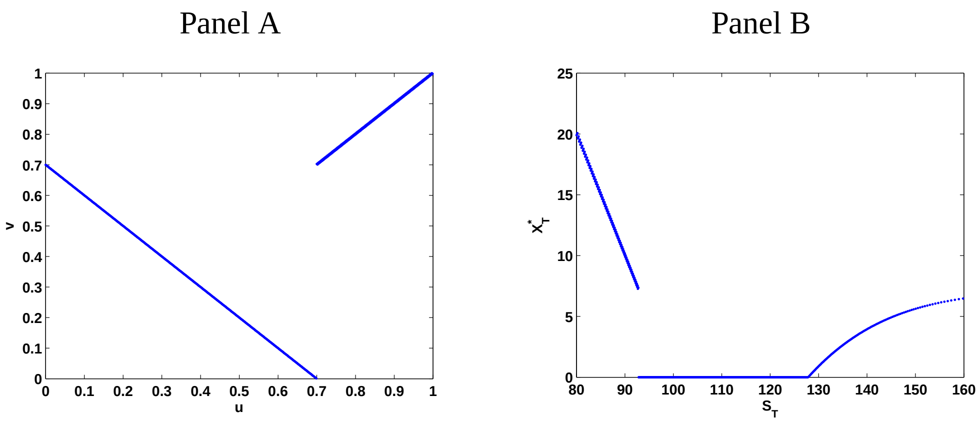

Figure 2: Panel A: Support of the lower bound of the copula between S_T and X_T. Panel B: Optimal Strategy under State-Dependent Constraint. Assumptions: $S_0 = 100$, $K = 100$, $\mu = 0.05$, $\sigma = 0.2$, $T = 1$, $r = 0.03$.

Note that Panels B of Figure 1 and Figure 2 both display an optimal strategy under probability constraints that is not non-decreasing with respect to the underlying S_T.

5. CONCLUSIONS

This paper presents optimal investment strategies in the presence of state-dependent constraints. Similarly as Bernard and Boyle (2010) the assumption is that one knows the cdf of terminal wealth and one wants to reach this objective cdf at the cheapest possible cost given some probability constraints. Investors with law-invariant preferences will solely invest in strategies that are non-decreasing in the underlying risky asset. In the presence of probability constraints, non-decreasing strategies in the risky asset are not necessarily optimal.

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CONTRIBUTED TALKS

GOOD-DEAL BOUNDS IN A REGIME-SWITCHING DIFFUSION MARKET

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Abstract

We consider the pricing of a maturity guarantee, which is equivalent to the pricing of a European put option, in a regime-switching market model. Regime-switching market models have been empirically shown to fit long-term stockmarket data better than many other models. However, since a regime-switching market is incomplete, there is no unique price for the maturity guarantee. We extend the good-deal pricing bounds idea to the regime-switching market model. This allows us to obtain a reasonable range of prices for the maturity guarantee, by excluding those prices which imply a Sharpe Ratio which is too high. The range of prices can be used as a plausibility check on the chosen price of a maturity guarantee.

1. INTRODUCTION

Maturity guarantees are a common addition to many life insurance policies. The policyholder is given a guarantee by the life insurance company that the proceeds of the policy at the maturity date is subject to a minimum value. Ensuring that the guarantee is properly valued is of concern to the life insurance company, since it is a potential threat to the solvency of the company. When investment market returns are depressed, the company's investments are reduced in value but this is precisely the time when the guarantee is likely to bite. Thus the financial burden of the guarantee on the company is exacerbated.

To begin to quantify the risks inherent in a maturity guarantee, we must value them appropriately. The primary aim of this paper is to obtain a method for the reasonable valuation of maturity guarantees within a model which is appropriate to the long-term nature of the guarantees. We ignore mortality and focus on the financial market model.

It is well-known that maturity guarantees have the same payoff as a European put option. To show this, denote the time to maturity of the insurance contract by T and suppose that the guaranteed benefit is amount K at time T. If the amount payable before the guarantee is applied equals $S(T)$, then the policyholder receives $\max[K, S(T)]$ at time T. This means that the insurance company is liable to pay an additional amount of $K - S(T)$ to the policyholder if the guarantee bites at the maturity date. We can write this mathematically as

$$\max[K - S(T), 0].$$

The above cost to the insurer is recognised as the payoff of a European put option with strike price K and time to maturity T. Thus valuing the maturity guarantee is equivalent to valuing a European put option.

To value the maturity guarantee, we assume a model of the stockmarket called a regime-switching market model. Regime-switching market models are a way of capturing discrete shifts in market behavior. These shifts could be due to a variety of reasons, such as changes in market regulations, government policies or investor sentiment. In particular, regime-switching market models are effective at capturing the long-term behaviour of the stock market (for example, see Hardy (2003, Chapter 3)). This is an extremely appealing feature if we are valuing maturity guarantees since often the guarantees are applied after many years.

Due to the regime-switching, the market is incomplete and hence there are no unique prices for derivatives. In fact, the range of possible prices for a particular derivative is too wide to be useful in practice. Various suggestions have been made on either how to choose a single price or how to obtain a more restricted, and therefore potentially more useful, range of prices. We focus in this paper on the latter because it is the market which ultimately decides the price and so we should take into account our uncertainty about what the market price will be. Therefore, we believe it is better to find a range of prices that the market-determined price might reasonably be expected to lie in, rather than determining a single price.

The idea that we build upon is that of the *good-deal bound*. This idea is due to Cochrane and Saá Requejo (2000) and is based on the Sharpe Ratio, which is the excess return on an investment per unit of risk. Their idea is to bound the Sharpe Ratios of all possible assets in the market and thus exclude Sharpe Ratios which are considered to be too large. The method of applying the good-deal bound gives a set of risk-neutral martingale measures which can be used to price options. This results in an *upper and lower good-deal pricing bound* on the prices of an option. The idea was streamlined and extended to models with jumps in Björk and Slinko (2006), and it is their approach that we follow in this paper.

The good-deal pricing bounds can be used by a life insurance company in the pricing of maturity guarantees in various ways. First, since ultimately a single price must be chosen so that an appropriate premium can be charged for the insurance contract, the good-deal pricing bounds can act as a plausibility check on the chosen single price. In this case, the life insurance company can select the bound on the Sharpe Ratio in accordance with their own risk preferences. Second, if we examine the change in the pricing bounds as the bound on the Sharpe Ratio changes, we see the sensitivity of the price of the maturity guarantee to changes in the market's price of risk. Third, the upper pricing bound could itself be used as the single price for the maturity guarantee.

The aim of this paper is to apply the good-deal bound idea to the pricing of derivatives in a regime-switching diffusion market. The paper is structured as follows. Section 2 details the regime-switching market model. In Section 3 we identify the set of equivalent martingale measures via the set of Girsanov kernel processes. In Section 4 the Sharpe Ratio of an arbitrary asset in the market is defined and we state the extended Hansen-Jagannathan bound. The definitions of the upper and lower good-deal bounds on the price of a derivative are in Section 5. The stochastic control approach that we use to find them is outlined in Section 6. A numerical example illustrating the upper and lower good-deal pricing bounds on a 10-year maturity guarantee (i.e. a 10-year European put option) is given in Section 7.

2. MARKET MODEL

We consider a regime-switching market model in which there is one traded asset and a risk-free asset.

Description of the market model

We consider a continuous-time financial market model on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where all investment takes place over a finite time horizon $[0, T]$, for a fixed $T \in (0, \infty)$. The probability space carries both a 1-dimensional standard Brownian motion W and a Markov chain α.

The information available to the investors in the market at time t is the history of the Markov chain and Brownian motion up to and including time t. Mathematically, this is represented by the filtration

$$\mathcal{F}_t := \sigma\{(\alpha(s), W(s)), s \in [0, t]\} \vee \mathcal{N}(\mathbb{P}), \quad \forall t \in [0, T],$$

where $\mathcal{N}(\mathbb{P})$ denotes the collection of all $\mathbb{P}$-null events in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that $\mathcal{F} = \mathcal{F}_T$.

The market is subject to regime-switching, as modelled by the continuous-time Markov chain α which takes values in a finite state space $I = \{1, \ldots, D\}$, for some integer $D \geq 2$. We assume that the Markov chain starts in a fixed state $i_0 \in I$, so that $\alpha(0) = i_0$, a.s. The Markov chain has a generator G, which is a $D \times D$ matrix $G = (g_{ij})_{i,j=1}^D$ with the properties $g_{ij} \geq 0$, for all $i \neq j$ and $g_{ii} = -\sum_{j \neq i} g_{ij}$. To avoid states where there are no transitions into or out of, we assume that $g_{ii} < 0$ for each state i.

Associated with each pair of distinct states (i, j) in the state space of the Markov chain is a point process, or counting process,

$$N_{ij}(t) := \sum_{0<s\leq t} \chi\left[\alpha(s_-) = i\right] \chi\left[\alpha(s) = j\right], \quad \forall t \in [0, T],$$

where χ denotes the zero-one indicator function. Define the intensity process

$$\lambda_{ij}(t) := g_{ij}\, \chi\left[\alpha(t_-) = i\right].$$

If we compensate $N_{ij}(t)$ by $\int_0^t \lambda_{ij}(s)ds$, then the resulting process

$$M_{ij}(t) := N_{ij}(t) - \int_0^t \lambda_{ij}(s)ds$$

is a martingale (see Rogers and Williams (2006, Lemma IV.21.12)). We refer to the set of martingales $\{M_{ij}; i, j \in I, j \neq i\}$ as *the* $\mathbb{P}$*-martingales of* α.

For simplicity, we consider a financial market that is built upon one traded asset, which we call the risky asset, and a risk-free asset. The risk-free rate of return in the market is denoted by the scalar stochastic process r and the risk-free asset's price process $S_0 = \{S_0(t), t \in [0,T]\}$ is governed by

$$\frac{dS_0(t)}{S_0(t)} = r(t)dt, \quad \forall t \in [0,T], \quad S_0(0) = 1. \tag{1}$$

The mean rate of return of the risky asset is denoted by the scalar stochastic process μ and the volatility process of the risky asset is denoted by the scalar stochastic process σ. The price process $S = \{S(t), t \in [0,T]\}$ of the risky asset is then given by

$$\frac{dS(t)}{S(t)} = \mu(t)dt + \sigma(t)dW(t), \quad \forall t \in [0,T], \tag{2}$$

with the initial value $S(0)$ being a fixed, strictly positive constant in $\mathbb{R}$.

We assume that the market parameters r, μ and σ are sufficiently regular to allow for the existence of a unique strong solution to (1) and (2). Furthermore, we assume that the volatility process σ of the risky asset is non-zero.

3. MARTINGALE MEASURES

In the regime-switching market model, while there is no arbitrage, the market is incomplete. This means that while equivalent martingale measures ("EMMs") exist, there is no unique one and hence we obtain a range of prices, called the no-arbitrage bounds, rather than a unique price for each derivative. The good-deal bound approach is a means of narrowing the no-arbitrage bounds, which are too wide to be useful in practice. The essential idea is to exclude those EMMs which imply a Sharpe Ratio that is too high. However, rather than dealing directly with the EMMs, we use instead the Girsanov kernel processes which generate EMMs.

3.1. The martingale condition

Given a Girsanov kernel process $(h, \boldsymbol{\eta})$, we can generate a corresponding measure $\mathbb{Q}$ by defining the likelihood process L as the process with dynamics

$$\frac{dL(t)}{L(t_-)} = h(t)dW(t) + \sum_{i=1}^{D} \sum_{\substack{j=1, \\ j \neq i}}^{D} \eta_{ij}(t) dM_{ij}(t), \quad \forall t \in [0,T],$$

and then construct the measure $\mathbb{Q}$ by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = L(t), \quad \text{on } \mathcal{F}_t.$$

Let $\mathbb{Q}$ be the measure generated by the Girsanov kernel process $(h, \boldsymbol{\eta})$. Consider an arbitrary asset in the market, with price process $\Pi = \{\Pi(t); t \in [0,T]\}$. Note that this asset is not restricted to the traded asset or the risk-free asset, but it could be any derivative or self-financing strategy based on them and the Markov chain α. The $\mathbb{P}$-dynamics of the asset's price process Π are of the form

$$\frac{d\Pi(t)}{\Pi(t_-)} = \mu^\Pi(t)dt + \sigma^\Pi(t)dW(t) + \sum_{i=1}^{D}\sum_{\substack{j=1,\\ j\neq i}}^{D} \gamma_{ij}^\Pi(t)dM_{ij}(t). \tag{3}$$

The processes μ^Π, σ^Π and $(\gamma_{ij}^\Pi)_{j\neq i}$ are suitably integrable and measurable with the condition, in order to avoid negative asset prices, that $\gamma_{ij}^\Pi(t) \geq -1$ for each $j \neq i$.

Applying a Girsanov theorem, we obtain the price dynamics Π of the arbitrarily chosen asset under the measure $\mathbb{Q}$:

$$\begin{aligned}\frac{d\Pi(t)}{\Pi(t_-)} = &\left(\mu^\Pi(t) + h(t)\sigma^\Pi(t) + \sum_{i=1}^{D}\sum_{\substack{j=1,\\ j\neq i}}^{D} \gamma_{ij}^\Pi(t)\eta_{ij}(t)\,\lambda_{ij}(t)\right)dt \\ &+ \sigma^\Pi(t)dW^{\mathbb{Q}}(t) + \sum_{i=1}^{D}\sum_{\substack{j=1,\\ j\neq i}}^{D} \gamma_{ij}^\Pi(t)dM_{ij}^{\mathbb{Q}}(t),\end{aligned}$$

in which $W^{\mathbb{Q}}$ is a $\mathbb{Q}$-Brownian motion and $M_{ij}^{\mathbb{Q}}$ is a $\mathbb{Q}$-martingale of the Markov chain α, for each $j \neq i$.

Proposition 3.1 Martingale condition. *The measure $\mathbb{Q}$ generated by the Girsanov kernel process $(h, \boldsymbol{\eta})$ is an equivalent martingale measure if and only if*

$$\eta_{ij}(t) > -1, \quad \forall j \neq i,$$

and for any asset in the market whose price process Π has $\mathbb{P}$-dynamics given by (3), we have

$$r(t) = \mu^\Pi(t) + h(t)\sigma^\Pi(t) + \sum_{i=1}^{D}\sum_{\substack{j=1,\\ j\neq i}}^{D} \gamma_{ij}^\Pi(t)\eta_{ij}(t)\,\lambda_{ij}(t), \quad \forall t \in [0, T]. \tag{4}$$

We refer to a Girsanov kernel process $(h, \boldsymbol{\eta})$, with $\boldsymbol{\eta} = \{\eta_{ij}; i, j \in I, j \neq i\}$, for which the generated measure $\mathbb{Q}$ is a martingale measure as an *admissible* Girsanov kernel process.

Remark 3.1 *From (4) we have the following economic interpretation of an admissible Girsanov kernel process $(h, \boldsymbol{\eta})$: the process $-h$ is the market price of diffusion risk and $-\eta_{ij}$ is the market price of regime change risk, for a jump in the Markov chain from state i to state j (i.e. a change from market regime i to market regime j).*

Suppose we are given a Girsanov kernel process $(h, \boldsymbol{\eta})$ for which the generated measure $\mathbb{Q}$ is an equivalent martingale measure. The price dynamics under $\mathbb{P}$ of the traded asset are as in (2). By Proposition 3.1, we must have that

$$r(t) = \mu(t) + h(t)\sigma(t), \quad \forall t \in [0, T].$$

This means that the market price of diffusion risk $-h$ is determined by the price dynamics of the traded asset.

4. THE SHARPE RATIO AND GOOD-DEAL BOUND

We define a Sharpe Ratio process for an arbitrarily chosen asset, with $\mathbb{P}$-dynamics as in (3). Broadly, the Sharpe Ratio is the excess return above the risk-free rate of the asset per unit of risk. We make this definition precise in our model. Define a *volatility process* ν for the asset as the process which satisfies

$$\nu^2(t) = |\sigma^\Pi(t)|^2 + \sum_{i=1}^{D} \sum_{\substack{j=1,\\ j\neq i}}^{D} |\gamma_{ij}^\Pi(t)|^2 \, \lambda_{ij}(t).$$

As μ^Π is the local mean rate of return of the asset under the measure $\mathbb{P}$, we define the *Sharpe Ratio process* (SR) for the arbitrarily-chosen asset as

$$(SR)(t) := \frac{\mu^\Pi(t) - r(t)}{\nu(t)}. \tag{5}$$

The Sharpe Ratio process depends on the chosen asset's price process. However, we seek a bound that applies to all assets' Sharpe Ratio processes. To do this, we use the extended Hansen-Jagannathan inequality, which is derived in Björk and Slinko (2006) and is an extended version of the inequality introduced by Hansen and Jagannathan (1991).

Lemma 4.1 (An extended Hansen-Jagannathan Bound) *For every admissible Girsanov kernel process $(h, \boldsymbol{\eta})$ and for any asset in the market whose price process Π has $\mathbb{P}$-dynamics given by (3) and, consequently, whose Sharpe Ratio process (SR) is given by (5), the following inequality holds:*

$$|(SR)(t)|^2 \leq |h(t)|^2 + \sum_{i=1}^{D} \sum_{\substack{j=1,\\ j\neq i}}^{D} |\eta_{ij}(t)|^2 \, \lambda_{ij}(t). \tag{6}$$

Proof. The proof follows that of Björk and Slinko (2006, Theorem A.1) and is therefore omitted. ■

The key idea is that, in order to restrict the set of equivalent martingale measures by way of the Sharpe Ratio, we use the Hansen-Jagannathan bound. Rather than bounding the Sharpe Ratios directly, we bound the right-hand side of (6) by a constant. We call the constant a *good-deal bound*.

Definition 4.1 *A good-deal bound is a constant $B \geq \sup_{t\in[0,T]} |h(t)|^2$, a.s.*

Remark 4.1 *A chosen good-deal bound B bounds the Sharpe Ratio process (SR) of any asset in the market as follows:*

$$|(SR)(t)|^2 \leq |h(t)|^2 + \sum_{i=1}^{D} \sum_{\substack{j=1,\\ j\neq i}}^{D} |\eta_{ij}(t)|^2 \, \lambda_{ij}(t) \leq B. \tag{7}$$

In other words, $|(SR)(t)| \leq \sqrt{B}$. The economic interpretation is that, under the good-deal bound approach, $\sqrt{B}$ and $-\sqrt{B}$ are the highest and lowest achievable instantaneous Sharpe Ratio in the market, respectively. However, in the regime-switching diffusion market, we see from (7) that the good-deal bound B is really a bound on the price $-\eta_{ij}$ of market price of regime change risk, since the price $-h$ of diffusion risk is determined by the traded asset.

5. THE GENERAL PROBLEM

We consider the valuation of a general contingent claim Z of the form

$$Z := \Phi(S(T), \alpha(T)),$$

for a deterministic, measurable function Φ. We consider the problem of finding the upper and lower good-deal bounds on the range of possible prices of the contingent claim Z.

Definition 5.1 *Suppose we are given a good-deal bound B and a positive constant $\epsilon \ll 1$. The* upper good-deal price process V^{upper} *for the bound B is the optimal value process for the control problem*

$$\sup_{(h,\boldsymbol{\eta})} E^{\mathbb{Q}} \left(e^{-\int_t^T r(\tau)d\tau} \Phi(S(T), \alpha(T)) \,\middle|\, \mathcal{F}_t \right), \tag{8}$$

where the predictable processes $(h, \boldsymbol{\eta})$ are subject to the constraints

$$h(t) = -\sigma^{-1}(t) \left(\mu(t) - r(t) \right), \tag{9}$$

$$\eta_{ij}(t) \geq -1 + \epsilon, \qquad \text{for } i, j = 1, \ldots, D, \quad j \neq i, \tag{10}$$

and

$$|h(t)|^2 + \sum_{i=1}^{D} \sum_{\substack{j=1,\\ j \neq i}}^{D} |\eta_{ij}(t)|^2 \lambda_{ij}(t) \leq B, \tag{11}$$

for all $t \in [0, T]$.

Definition 5.2 *The* lower good-deal price process V^{lower} *is defined as in Definition 5.1 except that "sup" in (8) is replaced by "inf".*

Remark 5.1 *The risk-neutral valuation formula in (8) implies that the local rate of return of the price process corresponding to the contingent claim $Z = \Phi(S(T), \alpha(T))$ equals the risk-free rate r under the measure $\mathbb{Q}$. The equality constraint (9) ensures that h is consistent with the market price of diffusion risk. Together with the constraint (10), these ensure that the measure $\mathbb{Q}$ generated by $(h, \boldsymbol{\eta})$ is an equivalent martingale measure, as in Proposition 3.1.*

Remark 5.2 *The only unknown in the constraints (9)-(11) is the market price of regime change risk $-\eta_{ij}(t)$. If we obtain wide good-deal pricing bounds for a derivative then this tells us that the choice of the market price of regime change risk $-\eta_{ij}(t)$ has a large impact on the derivative's price. Thus wide good-deal pricing bounds are a signal that we should explore additional ways of further restricting the possible values of the market price of regime change risk $-\eta_{ij}(t)$. This point is also made in Cochrane and Saá Requejo (2000).*

The goal is to calculate the upper and lower good-deal price processes and we do this using a stochastic control approach.

6. STOCHASTIC CONTROL APPROACH

To ensure that the Markovian structure is preserved under the martingale measure $\mathbb{Q}$, we need the condition that the maximum in (8) is taken over Girsanov kernel processes $(h, \boldsymbol{\eta})$ of the form

$$h(t) = h(t, S(t), \alpha(t_-)) \quad \text{and} \quad \eta_{ij}(t) = \eta_{ij}(t, S(t), \alpha(t_-)), \quad \forall j \neq i. \tag{12}$$

Under this condition, the optimal expected value in (8) can be written as $V^{\text{upper}}(t, S(t), \alpha(t_-))$ where the deterministic mapping $V^{\text{upper}} : [0, T] \times \mathbb{R}_+ \times I \to \mathbb{R}_+$ is known as the *optimal value function*. From general dynamic programming theory (for example, see Björk (2009, Chapter 19)), the optimal value function satisfies the following Hamilton-Jacobi-Bellman equation

$$\begin{aligned} \frac{\partial V}{\partial t} + \sup_{(h,\boldsymbol{\eta})} \left\{ \mathbb{A}^{(h,\boldsymbol{\eta})} V \right\} - rV = 0 \\ V(T, x, i) = \Phi(x, i), \end{aligned} \tag{13}$$

where the supremum in (13) is subject to the constraints (9) - (11). An application of Itô's formula (for example, see Protter (2005, Theorem V.18, page 278)) shows that the infinitesimal operator $\mathbb{A}^{(h,\boldsymbol{\eta})}$ is given by

$$\begin{aligned} \mathbb{A}^{(h,\boldsymbol{\eta})} V(t, x, i) = r(t, x, i) x \frac{\partial V}{\partial x}(t, x, i) + \frac{1}{2} \sigma^2(t, x, i) x^2 \frac{\partial^2 V}{\partial x^2}(t, x, i) \\ + \sum_{\substack{j=1, \\ j \neq i}}^{D} g_{ij} (1 + \eta_{ij}(t, x)) \left(V(t, x, j) - V(t, x, i) \right), \end{aligned} \tag{14}$$

for all $(t, x, i) \in [0, T] \times \mathbb{R}_+ \times I$.

Definition 6.1 *Given a good-deal bound B and a positive constant $\epsilon \ll 1$, the* upper good-deal function *for the bound B is the solution to the following boundary value problem*

$$\begin{aligned} \frac{\partial V}{\partial t}(t, x, i) + \sup_{(h,\boldsymbol{\eta})} \left\{ \mathbb{A}^{(h,\boldsymbol{\eta})} V(t, x, i) \right\} - r(t, x, i) V(t, x, i) = 0 \\ V(T, x, i) = \Phi(x, i), \end{aligned} \tag{15}$$

where $\mathbb{A}^{(h,\boldsymbol{\eta})}$ is given by (14) and the supremum is taken over all functions $(h, \boldsymbol{\eta})$ of the form (12) and satisfying

$$h(t, x, i) = -\sigma^{-1}(t, x, i) \left(\mu(t, x, i) - r(t, x, i) \right), \tag{16}$$

$$\eta_{ij}(t, x) \geq -1 + \epsilon, \qquad \text{for } j = 1, \ldots, D, \quad j \neq i, \tag{17}$$

and

$$|h(t, x, i)|^2 + \sum_{\substack{j=1, \\ j \neq i}}^{D} g_{ij} |\eta_{ij}(t, x)|^2 \leq B, \tag{18}$$

for all $(t, x, i) \in [0, T] \times \mathbb{R}_+ \times I$.

Definition 6.2 *The* lower good-deal function *is the solution to (15) but with the supremum replaced by an infimum.*

Rather than attempting to solve the partial integro-differential equation of (15) directly, we reduce it to two deterministic problems which we solve for each fixed triple $(t, x, i) \in [0, T] \times \mathbb{R}_+ \times I$. Moreover, as h is completely determined by (16), we need to solve only for the optimal $\boldsymbol{\eta}$. Therefore, given h satisfying (16), we do the following:

1. Solve the static optimization problem of finding the optimal $\bar{\boldsymbol{\eta}}$ which attains the supremum of $\mathbb{A}^{(h,\boldsymbol{\eta})}V(t, x, i)$ subject to the constraints (17) and (18).

2. Using the optimal $\bar{\boldsymbol{\eta}}$ found above, numerically find the solution V to

$$\frac{\partial V}{\partial t} + \mathbb{A}^{(h,\bar{\boldsymbol{\eta}})}V - rV = 0$$
$$V(T, x, i) = \Phi(x, i).$$

Examining (14), we see that the static optimization problem reduces to a problem of maximizing a linear function of $\eta_{ij}(t, x)$ (that is, maximizing the last term on the right-hand side of (14)), subject to a linear inequality constraint (17) and a quadratic inequality constraint (18). This can be solved using the Kuhn-Tucker method; a further discussion of the solution can be found in Donnelly (2011).

7. NUMERICAL EXAMPLE

Having applied the good-deal bound idea in a regime-switching diffusion market, we examine their usefulness by calculating the upper and lower good-deal pricing bounds for a 10-year European put option in a market where there are two regimes. This corresponds to calculating the pricing bounds on a 10-year maturity guarantee.

7.1. Market model

Suppose there are only two market regimes and time is measured in years. Assume the values of the market parameters given in Table 1 and take the generator of the Markov chain to be

$$G = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} -0.15 & 0.15 \\ 2 & -2 \end{pmatrix},$$

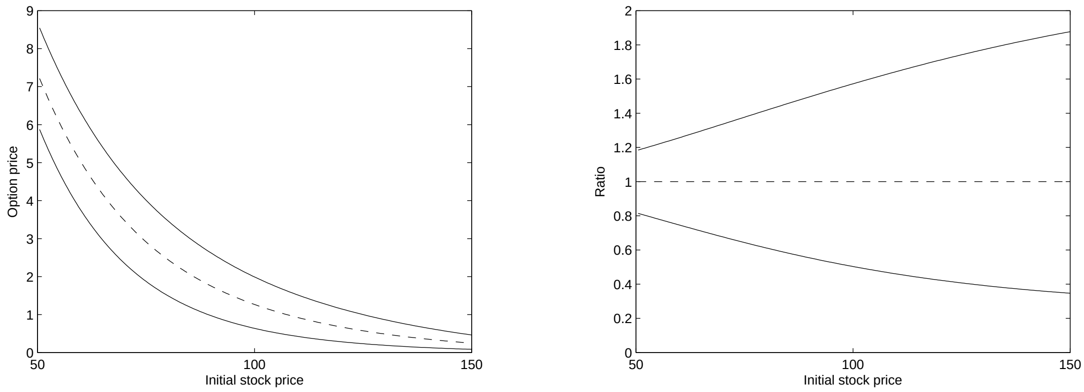
These figures are based loosely on the estimated parameters found in Hardy (2003, page 226) for a 2-state regime-switching model fitted to monthly return data from 1956 to 2001 from the FTSE-All Share Index, which covers over 98% of the U.K. stockmarket weighted by market capitalization. From the table, we see that regime 1 is a low volatility regime and regime 2 a high volatility regime. From the generator G, we see that the average time spent in regime 1 is nearly 7 years and the average time spent in regime 2 is 6 months. Thus the market is most of the time in the low volatility regime and only occasionally in the high volatility regime.

Regime i	$r(i)$	$\mu(i)$	$\sigma(i)$
1	0.06	0.15	0.12
2	0.06	-0.22	0.26

Table 1: Market parameters for the numerical example

7.2. Results

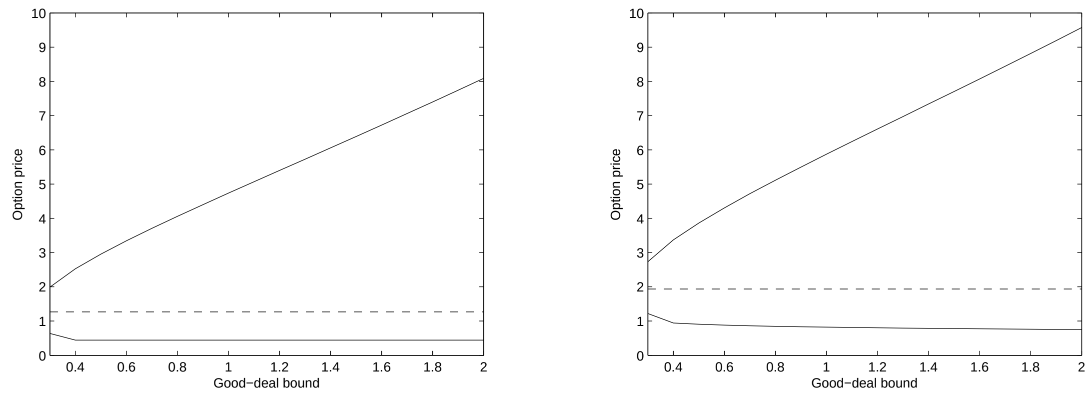
We fix the good-deal bound $B = 0.3$, which corresponds to the instantaneous Sharpe Ratio in the market being confined to the range $[-0.55, 0.55]$, and, considering all resulting bounds as open bounds, set $\epsilon = 0$. The upper and lower pricing bounds corresponding to $B = 0.3$ were calculated for a range of initial stock prices. The results are plotted as solid lines in Figure 1. The absolute prices are shown in Figure 1(a), with the dashed line corresponding to the price derived from the minimal martingale measure (the minimal martingale measure is determined by setting the market price of regime change risk $-\eta_{ij}(t)$ to be zero). Figure 1(b) shows the ratio of the bounds to the minimal martingale measure price. The latter figure demonstrates the impact of any assumption on the market price of the regime change risk; it shows the large variation of the pricing bounds, which allow for a non-zero price being assigned to the market price of regime change risk, from the minimal measure price, which assigns zero price to the market price of regime change risk.



(a) Absolute prices. (b) Relative prices.

Figure 1: The upper and lower good-deal pricing bounds for a 10-year European put option plotted against the initial stock price for a fixed good-deal bound $B = 0.3$. The strike price $K = 100$ and the initial market regime is regime 1. The upper solid line on each plot corresponds to the upper good-deal pricing bound, the lower solid line corresponds to the lower good-deal pricing bound and the dashed line corresponds to the minimal martingale measure price. The left plot shows the absolute prices and the right plot shows the bounds relative to the minimal martingale measure price. For example, the upper solid line in the right plot is obtained by dividing the upper good-deal bound price by the minimal martingale measure price.

Next we examine exactly how the pricing bounds change as we vary the good-deal bound B. We fix the initial stock price $S(0) = 100$ and $\epsilon = 0$ (again, the interpretation of the resulting pricing bounds is that they are open bounds), and calculate the pricing bounds for various choices of the good-deal bound B. These results are shown in Figure 2, with Figure 2(a) and 2(b) corresponding to the market starting in regime 1 and 2, respectively. Again, the minimal martingale measure prices are the dashed lines in the middle of each plot. As the good-deal bound B is increased, the permitted range of the instantaneous Sharpe Ratio in the market increases, and thus the pricing bounds increase. This demonstrates the sensitivity of the pricing bounds to the choice of the good-deal bound B. Notice that the lower bound in Figure 2(a) is constant for $B \geq 0.4$ because the inequality constraint (17) is binding at these values. Thus for $B \geq 0.4$ and starting in regime 1, the lower pricing bound is constant since it is calculated with a constant market price of regime change risk $-\eta_{ij}(t,x) = 1 - \epsilon = 1$.



(a) Starting in regime 1. (b) Starting in regime 2.

Figure 2: The upper and lower good-deal pricing bounds for a 10-year European put option with strike price $K = 100$ plotted against the good-deal bound. The initial stock price is $S(0) = 100$. The left plot assumes that the market is in regime 1 at time 0 and the right plot assumes that the market is in regime 2 at time 0. On both plots, the minimal martingale measure price is the dashed line.

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A COLLECTION OF RESULTS ON A FEYNMAN-KAC REPRESENTATION OF WEAK SOLUTIONS OF PIDES AND ON PRICING BARRIER AND LOOKBACK OPTIONS IN LEVY MODELS

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Abstract

Feynman–Kac formulas establish a fundamental link between conditional expectations and deterministic partial integro differential equations (PIDEs). In the context of option pricing in Lévy models, this relation has recently led to the development of various numerical methods to calculate prices via solving PIDEs. We give the precise link between certain conditional expectations and weak solutions of the corresponding PIDEs in Sobolev–Slobodeckii spaces. We apply the main result to price barrier options in (time-inhomogeneous) Lévy models and illustrate this by numerical results using a wavelet-Galerkin method.

We look at the characterization of option prices via solutions of PIDEs from two sides. In view of efficient numerical solutions, we concentrate on the formulation as parabolic equations in Sobolev–Slobodeckii spaces. Interpreting these equations as pseudo differential equations provides an appropriate access, when starting from Lévy models. A classification of Lévy processes according to their Fourier transforms is obtained.

The article provides a short description of parts of the results obtained in Glau (2010).

1. INTRODUCTION

The Feynman–Kac formula provides a link between conditional expectations and solutions of PDEs. In the context of Lévy processes, conditional expectations are linked to solutions of Partial Integro Differential Equations (PIDEs). In the last years this has lead to a remarkable development of algorithms to price options in Lévy models by solving PIDEs based on finite elements. In Matache et al. (2004), Matache et al. (2005b), Matache et al. (2005a), wavelet-Galerkin methods for pricing European and American options have been developed. The methods have been extended to multivariate models, see Reich et al. (2010), Winter (2009) and the references therein. Also standard finite element methods are efficiently used for pricing basket options in high dimensional models using dimension reduction techniques, see Hepperger (2010).

To study weak solutions of the PIDEs related to Lévy processes in terms of properties of the processes, in Glau (2010) and Glau (2011), the Sobolev index is defined and discussed in detail. While PIDEs are classified via their operators, due to the Lévy Khintchine formula, Lévy processes are completely described by their characteristic functions. Various classes of Lévy processes, as e.g. CGMY processes, are actually defined by specifying their characteristic function. On the other side the theory of weak solutions of partial differential equations relies on properties of the bilinear form which is associated with the operator of the equation. In the classical result about existence and uniqueness of weak solutions of evolution problems, both properties are related to the so-called Gårding and continuity inequalities for the bilinear form. The same relation is true for elliptic equations.

Within the framework of a time-inhomogeneous Lévy model for stock prices in Glau (2010) European, barrier, and lookback options are evaluated.

A Feynman–Kac representation for weak solutions of linear parabolic equations in Sobolev–Slobodeckii spaces is deduced. To adapt the result to the pricing of European options, we work with exponentially weighted Sobolev–Slobodeckii spaces. In order to characterize prices of barrier options by solutions of parabolic boundary value problems, we use a method known as “pénalisation du domaine“.

The result is applied to price barrier and lookback options numerically for a CGMY-model using a wavelet-Galerkin method.

2. THE MODEL AND BASIC NOTATION

We choose an exponential time-inhomogeneous Lévy model to describe stock prices. Time-inhomogeneous Lévy processes have proved to be useful for modeling financial derivatives, especially in the case of interest rate derivatives, see for example Eberlein et al. (2005), Eberlein and Özkan (2005), Eberlein and Kluge (2006), Eberlein and Koval (2006) and Eberlein and Liinev (2007).

In Glau (2010), a multivariate stock price model is considered. For the sake of brevity, we restrict ourselves in this article to the univariate case.

Let the stock price $S = (S_t)_{0\le t\le T}$ be given as

$$S_t = S_0\,\mathrm{e}^{L_t}, \qquad 0 \le t \le T \tag{1}$$

with a time-inhomogeneous Lévy process $L = (L_t)_{0\le t\le T}$ where $L_0 = 0$ and the local characteristics with respect to the truncation function h are $(b_t, \sigma_t, F_t)_{t\in[0,T]}$. Furthermore we model a riskless savings account S^0 via

$$S^0_t = \mathrm{e}^{\int_0^t r_s\,\mathrm{d}s}, \qquad 0 \le t \le T \tag{2}$$

with deterministic interest rate $r = (r_t)_{0\le t\le T}$. We additionally assume

$$\int_0^T \int_{\{x>1\}} \mathrm{e}^x\, F_s(\mathrm{d}x)\,\mathrm{d}s < \infty\,, \tag{3}$$

which is equivalent to $ES_t < \infty$ for $t \in [0,T]$.

We model under a risk-neutral measure. In view of assumption (3) this is the case iff the following drift condition is given

$$b_t = r_t - \frac{1}{2}\sigma_t^2 - \int_{\mathbb{R}} \big(\mathrm{e}^x - 1 - h(x) \big) F_t(\mathrm{d}x), \qquad 0 \le t \le T\,. \tag{4}$$

The infinitesimal generator $\mathcal{G}$ of the time-inhomogeneous Lévy process L is given by

$$\begin{aligned} \mathcal{G}_t f(x) &= \frac{\sigma_t}{2} f''(x) + b_t f'(x) \\ &\quad + \int_{\mathbb{R}} \big(f(x+y) - f(x) - h(y) f'(x) \big) F_t(\mathrm{d}y) \end{aligned} \tag{5}$$

for $f \in C_0^2(\mathbb{R})$.

We define the operator $\mathcal{A} = -\mathcal{G}$. It turns out that $\mathcal{A}$ is a pseudo differential operator (PDO) with symbol A i.e.

$$\mathcal{A}\, u(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{e}^{-i\xi x} A(\xi) \hat{u}(\xi)\, \mathrm{d}\xi$$

for all Schwartz-functions u. The symbol of the process L satisfies

$$A_t(\xi) = \frac{\sigma_t}{2}\xi^2 + i b_t \xi - \int_{\mathbb{R}} \big(\mathrm{e}^{-i\xi y} - 1 - i h(y)\xi \big) F_t(\mathrm{d}y) = -\theta_t(-\xi) \tag{6}$$

with

$$E\, \mathrm{e}^{i\xi L_t} = \mathrm{e}^{-\int_0^t A_s(-\xi)\, \mathrm{d}s}$$

for $0 \le t \le T$.

3. PRELIMINARIES

Let us consider a European option with payoff

$$\widetilde{g}(S_T) = g(L_T)$$

at maturity T where $x \mapsto \mathrm{e}^{\eta x} g(x)$ is in $L^2(\mathbb{R})$ i.e. $g \in L^2_\eta(\mathbb{R})$. For example, for a call option the payoff is $g(x) := S_0(\mathrm{e}^x - K/S_0)^+$ and we have $g\, \mathrm{e}^{\eta \cdot} \in L^2(\mathbb{R})$ for every $\eta < -1$.

Moreover we study barrier options with payoff

$$g(L_T) \mathbb{1}_{\{T < \tau_{\overline{D}}\}}$$

at maturity T, where D is an open set in $\mathbb{R}$ and $\tau_{\overline{D}}$ denotes the first exit time of the process L from the closure $\overline{D}$ of D,

$$\tau_{\overline{D}} = \inf\{ s \ge 0 | L_s \notin \overline{D} \}\,.$$

We characterize fair prices of those options by weak solutions of PIDEs in Sobolev–Slobodeckii spaces.

A function u belongs to the Sobolev–Slobodeckii space $H^s(\mathbb{R})$ for a certain $s \in \mathbb{R}$ iff $u \in L^1_{\mathrm{loc}}(\mathbb{R})$ and u has a Fourier transform $\mathcal{F}(u)$ in the weak sense and

$$\int (1 + |\xi|^2)^s |\mathcal{F}(u)(\xi)|^2 \, \mathrm{d}\xi < \infty \, .$$

Furthermore we define the weighted Sobolev–Slobodeckii spaces $H^s_\eta(\mathbb{R})$ for $s > 0$ and $\eta \in \mathbb{R}$ as the spaces of functions $u \in L^2_\eta(\mathbb{R})$, where $x \mapsto u(x)\,\mathrm{e}^{\eta x}$ belongs to $H^s(\mathbb{R})$. Moreover for $s \in \mathbb{R}$ the dual space of $H^s_\eta(\mathbb{R})$ is isomorphic to $H^{-s}_\eta(\mathbb{R})$.

We consider the following additional assumptions which were introduced in Glau (2010), p. 154. Let $\alpha \in [1, 2]$ and $\eta \in \mathbb{R}$. By $U_{-\eta}$ we denote the strip $U_{-\eta} := \mathbb{R} - i\,\mathrm{sgn}(\eta)[0, |\eta|)$ in the complex plane for $\eta \neq 0$, and $U_0 := \mathbb{R}$.

(A1) Assume

$$\int_0^T \int_{\{|x|>1\}} \mathrm{e}^{-\eta x} \, F_s(\mathrm{d}x) \, \mathrm{d}s < \infty \, .$$

(A2) There exists a constant $C_1 > 0$ with

$$\big|A_t(z)\big| \leq C_1 \big(1 + |z|\big)^\alpha$$

for all $z \in U_{-\eta}$ and for all $t \in [0, T]$.

(A3) There exist constants $C_2 > 0$ and $C_3 \geq 0$, such that for a certain $0 \leq \beta < \alpha$

$$\Re\big(A_t(z)\big) \geq C_2 \big(1 + |z|\big)^\alpha - C_3 \big(1 + |z|\big)^\beta$$

for all $z \in U_{-\eta}$ and for all $t \in [0, T]$.

(A4) The mapping $t \mapsto \Re(A_t(\xi - i\eta))$ is continuous and piecewise continuously differentiable with

$$\big|\partial_t \Re(A_t(\xi - i\eta))\big| \leq C_4 \big(1 + |\xi|\big)^\alpha$$

for every $t \in (0, T)$, where $t \mapsto A_t(\cdot - i\eta)$ is differentiable.

The symbol of a Lévy process does not depend on time and hence condition (A4) is irrelevant for Lévy processes. Conditions (A1)–(A3) are for example satisfied for CGMY-processes with parameters $C,\ G,\ M > 0$ and $Y \in [1, 2)$ with $\alpha = Y$ and $\eta \in (-G, M)$.

For Lévy processes with Brownian part, conditions (A2) and (A3) can be verified for $\alpha = 2$ and those $\eta \in \mathbb{R}$ that satisfy assumption (A1). In particular the Brownian motion (with drift) itself satisfies the assumptions for every $\eta \in \mathbb{R}$. See Glau (2011) and Glau (2010) for a detailed discussion of examples.

Condition (A1) about the existence of exponential moments is equivalent to $E\,\mathrm{e}^{-\eta' L_t} < \infty$ for every $0 \leq t \leq T$ and η' with $|\eta'| \leq |\eta|$ and $\mathrm{sgn}(\eta') = \mathrm{sgn}(\eta)$. Symbols A that satisfy this assumption have a continuous extension to $\overline{U_{-\eta}}$ that is analytic in the interior $\mathring{U}_{-\eta}$ of $U_{-\eta}$.

Under assumptions (A2) and (A3) the bilinear form a, given by

$$a(t; \varphi, \psi) := \big(\mathcal{A}_t \varphi\big)(\psi)$$

for all continuous functions φ, ψ with compact support, is continuous and satisfies a Gårding inequality with respect to the norm $\|\cdot\|_{\alpha/2,\eta}$ of the Hilbert space $H^{\alpha/2}_\eta(\mathbb{R})$. This means the bilinear form a has a unique extension to $[0,T]\times H^{\alpha/2}_\eta(\mathbb{R})\times H^{\alpha/2}_\eta(\mathbb{R})$ which is continuous in the sense

$$\big|a(t;u,v)\big| \le c_1\|u\|_{H^{\alpha/2}_\eta}\|v\|_{H^{\alpha/2}_\eta} \qquad \big(u,\, v\in H^{\alpha/2}_\eta(\mathbb{R})\big)$$

for every $t\in[0,T]$ with a constant $c_1>0$ and satisfies the Gårding inequality

$$\Re\big(a(t;u,u)\big) \ge c_2\|u\|^2_{H^{\alpha/2}_\eta} - c_3\|u\|^2_{L^2_\eta} \qquad \big(u,\, v\in H^{\alpha/2}_\eta(\mathbb{R})\big)$$

with constants $c_2>0$ and $c_3\ge 0$. See (Glau 2010, Theorem II.7 and II.9).

Let us emphasize that both conditions, the continuity and the Gårding condition, are required uniformly in time.

In this case the PIDEs of the form

$$\begin{aligned}\dot u + \mathcal{A}_t u &= f\\ u(0) &= g\,,\end{aligned}$$

with f in the dual space of the Sobolev–Slobodeckii space $H^{\alpha/2}_\eta(\mathbb{R})$ and with initial condition $g\in L^2_\eta(\mathbb{R})$ possess a unique solution $u\in W^1\big(0,T;H^{\alpha/2}_\eta(\mathbb{R});L^2_\eta(\mathbb{R})\big)$. We then have that $u\in L^2\big(0,T;H^{\alpha/2}_\eta(\mathbb{R})\big)$ with a derivative $\dot u$ with respect to time in the weak sense that satisfies $\dot u\in L^2\big(0,T;H^{-\alpha/2}_\eta(\mathbb{R})\big)$.

The space $W^1\big(0,T;H^{\alpha/2}_\eta(\mathbb{R}),L^2_\eta(\mathbb{R})\big)$ consists of those functions $u\in L^2\big(0,T;H^{\alpha/2}_\eta(\mathbb{R})\big)$ that have a derivative with respect to time $\dot u$ in a distributional sense that belongs to the space $L^2\big(0,T;\big(H^{\alpha/2}_\eta(\mathbb{R})\big)^*\big)$. For a Hilbert space H, the space $L^2\big(0,T;H\big)$ denotes the space of functions $u:[0,T]\to H$, that are weakly measurable and that satisfy $\int_0^T\|u(t)\|^2_H\,\mathrm{d}t<\infty$. For the definition of weak measurability and for a detailed introduction of the space $W^1\big(0,T;H^{\alpha/2}_\eta(\mathbb{R}),L^2_\eta(\mathbb{R})\big)$ that relies on the Bochner integral, we refer to the book of Wloka (1987).

4. SOBOLEV INDEX OF A LEVY PROCESS

Let us briefly discuss the definition of a Sobolev index for Lévy processes. The index is discussed in detail in Glau (2011).

Definition 4.1 *Let $\mathcal{A}$ be a PDO with symbol A. If there exists a real number $\alpha\in(0,2]$ such that for all $\xi\in\mathbb{R}$*

$$\begin{aligned}\big|A(\xi)\big| &\le C_1\,(1+|\xi|^2)^{\alpha/2} && \textit{(continuity condition), and}\\ \Re\big(A(\xi)\big) &\ge C_2|\xi|^\alpha - C_3\,(1+|\xi|^2)^{s/2} && \textit{(Gårding condition)}\end{aligned}$$

with $0<s<\alpha$ and constants $C_1,\,C_3\ge 0$ and $C_2>0$, then we call α the Sobolev index *of the symbol A. If L is a Lévy process with symbol A with Sobolev index α, we call α the* Sobolev index of the Lévy process *L.*

Not every Lévy process has a Sobolev index, but for a wide range of Lévy processes there is a Sobolev index, and if the index is smaller than 2, it is equal to the Blumenthal-Getoor index. The Sobolev index exists e.g. for CGMY and generalized hyperbolic (GH) processes and for Lévy processes with a Lévy measure which has a Lebesgue density.

The following relationship to parabolic equations is derived.

Theorem 4.1 *Let $\mathcal{A}$ be a PDO whose symbol A has a Sobolev index α for some $\alpha > 0$. Then the parabolic equation*

$$\begin{aligned} \partial_t u + \mathcal{A}\, u &= f \\ u(0) &= g\,, \end{aligned} \tag{7}$$

for $f \in L^2\big(0,T;H^{-\alpha/2}(\mathbb{R})\big)$ and initial condition $g \in L^2(\mathbb{R})$ has a unique weak solution u in the space $W^1\big(0,T;H^{\alpha/2}(\mathbb{R}),L^2(\mathbb{R})\big)$.

5. PIDE TO PRICE EUROPEAN OPTIONS

Let Π_t^g denote the fair price of a European option with payoff $g(L_T)$ at maturity T. Using the Markov property of the process L we derive

$$\Pi_t^g = E\Big(g(L_T)\,\mathrm{e}^{-\int_t^T r_s\,\mathrm{d}s}\,\Big|\mathcal{F}_t\Big) = E\Big(g(L_T)\,\mathrm{e}^{-\int_t^T r_s\,\mathrm{d}s}\,\Big|L_t\Big) = u(T-t,L_t)\,.$$

The following theorem is deduced in (Glau 2010, Theorem V.2), see also (Glau 2010, Satz II.13 and Theorem IV.9).

Theorem 5.1 *Let us assume (A1)–(A4) for an $\alpha \in [1,2]$ and an $\eta \in \mathbb{R}$ with $g \in L^2_\eta(\mathbb{R})$. The fair price Π_t^g of the option at time $t \in [0,T]$ is given by*

$$u(T-t,L_t) = E\Big(g(L_T)\,\mathrm{e}^{-\int_t^T r_s\,\mathrm{d}s}\,\Big|L_t\Big)$$

where the function $u \in W^1\big(0,T;H_\eta^{\alpha/2}(\mathbb{R});L^2_\eta(\mathbb{R})\big)$ is the unique solution of the parabolic equation

$$\begin{aligned} \partial_t u + \mathcal{A}_t u + r_t u &= 0 \\ u(0) &= g\,. \end{aligned}$$

Furthermore $u \in C^1\big(0,T;H_\eta^m(\mathbb{R})\big)$ for every $m \in \mathbb{N}$ and the following holds:

$$\mathcal{F}\Big(\,\mathrm{e}^{\eta}\,u\Big)(t) = \hat{g}(\cdot - i\eta)\,\mathrm{e}^{-\int_0^t A_s(\cdot - i\eta)\,\mathrm{d}s}\,\mathrm{e}^{-\int_t^T r_s\,\mathrm{d}s}\;. \tag{8}$$

Equation (8) coincides with the so-called convolution formula derived in Raible (2000), and in Carr and Madan (1999) for the case of a call option.

To summarize: the prices of European options are given in terms of weak solutions of PIDEs. The interpretation as pseudo differential equation corresponds to the convolution method, where the option price is written as a convolution and is represented via Fourier transforms.

6. PIDE TO PRICE BARRIER OPTIONS

We price barrier options with payoff

$$g(L_T)\mathbb{1}_{\{T<\tau_{\overline{D}}\}}$$

where $\tau_{\overline{D}}$ denotes the first exit time of the process L from $\overline{D}$.

We derive a stochastic representation of the parabolic equation of the form

$$\begin{aligned} \dot{u} + \mathcal{A}_t u &= f \qquad \text{in } D \subset \mathbb{R} \\ u(0) &= g\,, \end{aligned}$$

$u \equiv 0$ in D^c. The precise mathematical formulation of the equation is achieved by introducing the (weighted) Sobolev–Slobodeckii space $\widetilde{H}^{\alpha/2}_{\eta}(D)$ which is the subspace of those functions $u \in H^{\alpha/2}_{\eta}(\mathbb{R})$ that are vanishing on D^c, the complement of D.

We obtain the Feynman–Kac formula using a method called 'pénalisation du domaine', that is outlined in (Glau 2010, Kapitel III), see also (Glau 2010, Theorem V.4 and Theorem IV.9).

Theorem 6.1 *Let the assumptions (A1)–(A4) be satisfied for an $\alpha \in [1,2]$ and an $\eta \in \mathbb{R}$ with $g \in L^2_\eta(\mathbb{R})$. The fair price of the barrier option at time $t \in [0,T]$ is given by $\Pi_t = u(T-t, L_t)\mathbb{1}_{t<\tau_{\overline{D}}}$ where*

$$u(T-t, L_t) = E\big(g(L_T)\mathbb{1}_{\{T<\tau_{t,\overline{D}}\}}\, \mathrm{e}^{-\int_t^T r_s\,\mathrm{d}s}\,\big|\mathcal{F}_t\big)$$

with $\tau_{t,\overline{D}} = \inf\{s \geq t | L_s \notin \overline{D}\}$. The function u is the unique weak solution in the space $u \in W^1\big(0,T;\widetilde{H}^{\alpha/2}_\eta(D); L^2_\eta(D)\big)$ of

$$\begin{aligned} \partial_t u + \mathcal{A}_t u + r_t u &= 0 \\ u(0) &= g\,. \end{aligned}$$

For a digital up-and-out barrier option with barrier $H = S_0\,\mathrm{e}^B$ for example, the initial function is chosen as $g(x) = \mathbb{1}_{(-\infty,B)}(x)$. The price of the digital barrier option is

$$\Pi_t^{\mathrm{digi},B} = u^{\mathrm{digi},B}(T-t, L_t)\mathbb{1}_{\{t<\tau_{\overline{(-\infty,B)}}\}}\,,$$

where $u^{\mathrm{digi},B}$ is the unique solution $u \in W^1\big(0,T;\widetilde{H}^{\alpha/2}_\eta(-\infty,B); L^2_\eta(-\infty,B)\big)$ of the parabolic boundary value problem

$$\begin{aligned} \partial_t u + \mathcal{A}_t u + r_t u &= 0 \\ u(0) &= \mathbb{1}_{(-\infty,B)}\,, \end{aligned} \tag{9}$$

compare (Glau 2010, Korollar V.5).

7. PRICING LOOKBACK OPTIONS VIA PIDES

Let the process L be a Lévy process and let the assumptions (A1)–(A4) be satisfied for an $\alpha \in [1,2]$ and a certain $\eta > 0$. In this case, the price of the digital barrier option at time $t = 0$ corresponds to the distribution function $F^{\overline{L}_T}$ of the supremum $\overline{L}_T = \sup_{0\le t\le T} L_t$, i.e.

$$F^{\overline{L}_T}(x) = P(\overline{L}_T < x) = u^{\mathrm{digi},0}(T,-x)\,.$$

The fair price

$$V_0 = V_0(S_0) = \mathrm{e}^{-\int_0^T r_s\,\mathrm{d}s}\, E\Big(\sup_{0\le t\le T} S_t - K\Big)^+$$

of the lookback option at time 0 is then given by

$$V_0(S_0) = S_0\,\mathrm{e}^{-\int_0^T r_s\,\mathrm{d}s}\left(\int_{k-\log(S_0)}^{\infty}\big(1-u^{\mathrm{digi},0}(T,-x)\big)\,\mathrm{e}^x\,\mathrm{d}x + (1-K/S_0)^+\right), \qquad (10)$$

where the function $u^{\mathrm{digi},0}$ is the unique solution of the parabolic boundary value problem (9) for $B=0$. This is the basis for deriving a PIDE to price the lookback option in (Glau 2010, Kapitel VI.2.2). More precisely, a PIDE for the integrand in equation (10) is derived and solved numerically using a wavelet-Galerkin scheme, see (Glau 2010, p. 187-190).

8. NUMERICAL EVALUATION

For the numerical evaluation we choose a Lévy model with a CGMY process as driving process. We calculate the option prices using a wavelet-Galerkin method. The main part of the method was developed by Schwab et al., see e.g. von Petersdorff and Schwab (2003), Matache et al. (2004), Matache et al. (2005b). They also provided a large part of the code. For a description of the specific algorithm see (Glau 2010, Kapitel VI.1).

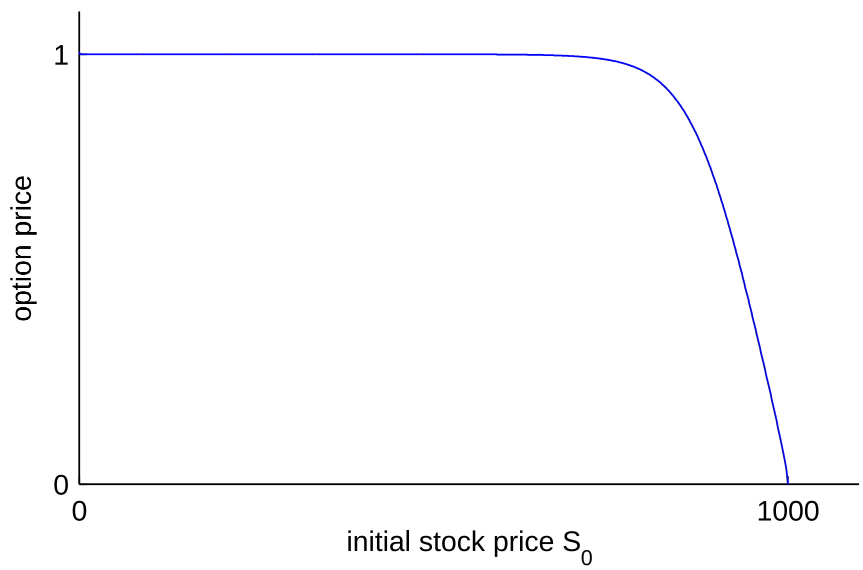

Figure 1: Price of a digital barrier option with Barrier at 1000 and maturity 1 year in a CGMY model.

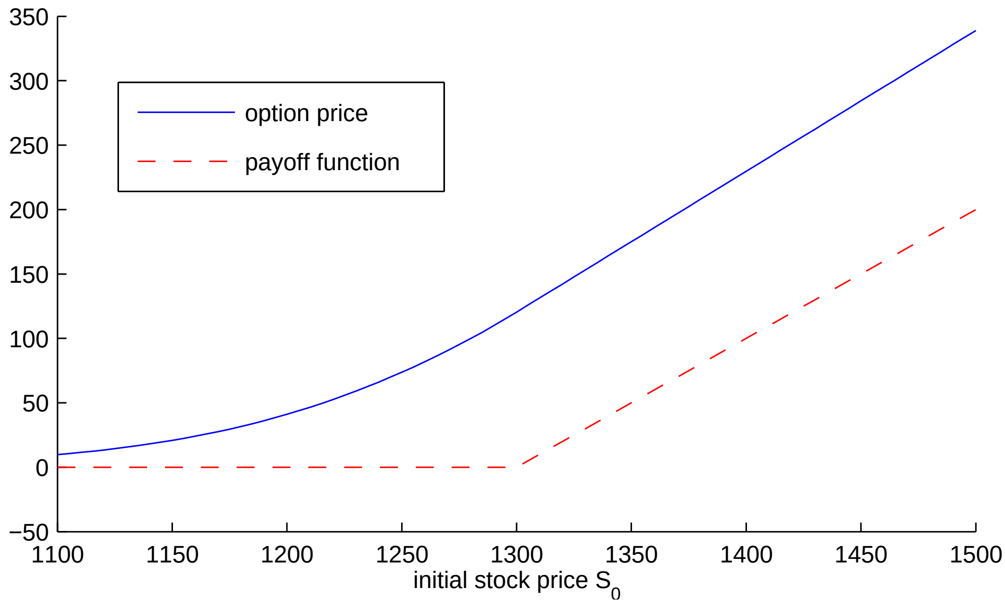

Figure 2: The figure shows the price of the lookback option and the payoff function $(S_0 - K)^+$ at maturity 1 year with strike 1300.

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THE GENERALIZED αVG MODEL

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Abstract

This paper extends the αVG model (Semeraro 2008) by relaxing the constraints on the Gamma subordinator parameters, leading to marginal characteristic functions of the asset log-returns which become a function of the whole parameter set. Hence, the calibration of this generalized model does not require anymore any correlation fit, which might turn out to be a significant advantage in practice since the risk-neutral calibration of the correlation requires the existence of a liquid market for multivariate derivatives which is nowadays pretty rare. Moreover, the volatility of the log-returns depends on both the common and idiosyncratic subordinator settings, and not only on the idiosyncratic one as under the original model, which makes the generalized model more in line with the empirical evidence of the presence of both an idiosyncratic and a common component in the business time.

1. INTRODUCTION

The use of a time-changed Brownian motion in finance was first proposed by Clark to model cotton future prices (Clark 1973). His pioneer work was motivated by the fact that the information flow directly affects the evolution of the price through time. More precisely, when the amount of available information is low, the trading is slow and the price process evolves slowly and the other way around. Since then the concept of business clock has been widely considered in the financial literature, first to model univariate stock price processes (Ané and Geman (2000), Carr et al. (2003), Madan and Senata (1990)), before being extended to the multivariate setting. Madan and Senata (1990) first proposed to subordinate a multivariate Brownian motion by an univariate Gamma time change. However, the uniqueness of the business clock makes impossible to capture independency of the stock log-returns. Hence, Semeraro (2008) proposed the so-called αVG model which rests on a multivariate subordinator process composed of the weighted sum of two independent Gamma processes: an idiosyncratic and a common component. Later, Luciano and Semeraro (2010) extended the αVG model to other Lévy distributions by considering other subordinators. This class

of multivariate models was motivated by the empirical work of Lo and Wang (2000) which gives evidence for the presence of a significant common component in the trading volume and by the study of Harris (1986) which shows that the distribution of the information flow is not identical for all securities. In the original setting, Luciano and Semeraro imposed some restrictions on the subordinator parameters such that the subordinator follows the same distribution as its two components, leading to marginal log-return processes of a particular Lévy type. Under this restricted setting, the marginal characteristic functions become independent of the common subordinator setting which affects only the dependence structure of the asset log-returns. This might lead to two undesired features in practice. First, the risk-neutral calibration of the common subordinator parameters requires liquid multivariate derivative quotes which are often unavailable. Secondly, the variance and therefore the volatility of the asset log-returns turn out to be independent of the common subordinator setting. Since the volatility level is directly related to the trading activity, the conditions imposed on the time change parameters imply that the trading activity does not depend on the common component of the business clock, but only on the idiosyncratic one.
If the marginal class is not a desired feature, the model can be extended by relaxing the constraints imposed on the subordinator parameters. The such obtained generalized αVG model belongs to the class of exponential Lévy model, although the particular underlying Lévy distribution is not known anymore. We will show that the marginal characteristic functions and consequently also the volatility of the asset log-returns then depend on both the idiosyncratic and common subordinator settings and more specifically on the whole set of parameters. Hence the calibration of the generalized αVG model does not require the existence of actively traded multivariate derivatives anymore.

2. THE αVG MODEL

Under the αVG model, the N-dimensional stock price process satisfies:

$$\mathbf{S}_t = \begin{pmatrix} S_t^{(1)} \\ S_t^{(2)} \\ \vdots \\ S_t^{(N)} \end{pmatrix} = \begin{pmatrix} S_0^{(1)} \exp\left((r - q_1 + \omega_1)t + Y_t^{(1)}\right) \\ S_0^{(2)} \exp\left((r - q_2 + \omega_2)t + Y_t^{(2)}\right) \\ \vdots \\ S_0^{(N)} \exp\left((r - q_N + \omega_N)t + Y_t^{(N)}\right) \end{pmatrix} = \begin{pmatrix} \frac{S_0^{(1)} \exp\left((r-q_1)t+Y_t^{(1)}\right)}{\mathbb{E}\left[\exp\left(Y_t^{(1)}\right)\right]} \\ \frac{S_0^{(2)} \exp\left((r-q_2)t+Y_t^{(2)}\right)}{\mathbb{E}\left[\exp\left(Y_t^{(2)}\right)\right]} \\ \vdots \\ \frac{S_0^{(N)} \exp\left((r-q_N)t+Y_t^{(N)}\right)}{\mathbb{E}\left[\exp\left(Y_t^{(N)}\right)\right]} \end{pmatrix},$$

where $S_0^{(i)}$ is the spot price of the ith underlying, r is the risk-free interest rate, q_i denotes the dividend yield of the ith stock and $\omega = (\omega_1, \omega_2, \ldots, \omega_N)^T$ is the mean correcting vector making the model risk-neutral. The process $\mathbf{Y} = \{\mathbf{Y}_t, t \geq 0\}$ is a N-dimensional time-changed Brownian motion:

$$\mathbf{Y}_t = \begin{pmatrix} Y_t^{(1)} \\ Y_t^{(2)} \\ \vdots \\ Y_t^{(N)} \end{pmatrix} = \begin{pmatrix} \theta_1 G_t^{(1)} + \sigma_1 W^{(1)}_{G_t^{(1)}} \\ \theta_2 G_t^{(2)} + \sigma_2 W^{(2)}_{G_t^{(2)}} \\ \vdots \\ \theta_N G_t^{(N)} + \sigma_N W^{(N)}_{G_t^{(N)}} \end{pmatrix}, \quad (1)$$

where $W^{(i)}, i = 1, \ldots, N$ are independent standard Brownian motions and where the subordinators $G_t^{(i)}$'s are the weighted sum of two Gamma processes, one idiosyncratic and one common process:

$$\mathbf{G}_t = \begin{pmatrix} G_t^{(1)} \\ G_t^{(2)} \\ \vdots \\ G_t^{(N)} \end{pmatrix} = \begin{pmatrix} X_t^{(1)} + \alpha_1 Z_t \\ X_t^{(2)} + \alpha_2 Z_t \\ \vdots \\ X_t^{(N)} + \alpha_N Z_t \end{pmatrix},$$

where $\alpha_i > 0$, $Z_1 \sim$ **Gamma**$(c_1, c_2), c_1, c_2 > 0$ and $X_1^{(i)} \sim$ **Gamma**(a_i, b_i), $a_i, b_i > 0$ are independent random variables and are independent on the $W^{(i)}$'s.

- **The Gamma process**

  The characteristic function of the Gamma distribution **Gamma**(a, b) with parameters $a > 0$, $b > 0$ is given by:

$$\phi_{\text{Gamma}}(u; a, b) = \left(1 - \frac{\mathrm{i}u}{b}\right)^{-a}.$$

  The Gamma process $X = \{X_t, t \geq 0\}$ is a Lévy process such that X_t follows a **Gamma**(at, b) distribution. The Gamma distribution satisfies the following scaling property: if $X \sim$ **Gamma**(a, b) then $cX \sim$ **Gamma**$(a, b/c), c > 0$. Moreover, the sum of independent Gamma random variables with the same parameter b is also Gamma distributed: if $X_i \sim$ **Gamma**(a_i, b), $i = 1, \ldots, N$ are N independent random variables then $\sum_{i=1}^{N} X_i \sim$ **Gamma** $(\sum_{i=1}^{N} a_i, b)$. The first four moments of the Gamma distribution are given in Table 1.

	Gamma(a, b)
mean	$\frac{a}{b}$
variance	$\frac{a}{b^2}$
skewness	$\frac{2}{\sqrt{a}}$
kurtosis	$3\left(1 + \frac{2}{a}\right)$

Table 1: Characteristics of the Gamma distribution.

- **The Variance Gamma process**

  The characteristic function of the Variance Gamma distribution $\mathbf{VG}(\sigma,\nu,\theta)$ with parameters $\sigma>0$, $\nu>0$ and $\theta\in\mathbb{R}$ is given by:

$$\phi_{\text{VG}}(u;\sigma,\nu,\theta)=\left(1-\mathrm{i}u\theta\nu+\frac{u^2\sigma^2\nu}{2}\right)^{\frac{-1}{\nu}},\quad u\in\mathbb{R}.$$

  The Variance Gamma process $X=\{X_t,t\geq 0\}$ is a Lévy process such that X_t follows a $\mathbf{VG}(\sqrt{t}\sigma,\frac{\nu}{t},\theta t)$ distribution. The VG distribution satisfies the following scaling property: if $X\sim\mathbf{VG}(\sigma,\nu,\theta)$ then, for $c>0$, $cX\sim\mathbf{VG}(c\sigma,\nu,c\theta)$. The first four moments of the VG distribution are given in Table 2.

	$\mathbf{VG}(\sigma,\nu,\theta)$
mean	θ
variance	$\sigma^2+\nu\theta^2$
skewness	$\frac{\theta\nu\left(3\sigma^2+2\nu\theta^2\right)}{\left(\sigma^2+\nu\theta^2\right)^{\frac{3}{2}}}$
kurtosis	$3\left(1+2\nu-\frac{\nu\sigma^4}{(\sigma^2+\nu\theta^2)^2}\right)$

Table 2: Characteristics of the Variance Gamma distribution.

  A VG(σ,ν,θ) process can be seen as a Gamma time-changed Brownian motion with drift:

$$X_t^{\mathbf{VG}}=\theta G_t+\sigma W_{G_t}$$

  where $G=\{G_t,t\geq 0\}$ is a Gamma process with parameters $a=b=\frac{1}{\nu}$ and $W=\{W_t,t\geq 0\}$ is a standard Brownian motion.

3. The GENERALIZED αVG MODEL

The characteristic function of the process $\mathbf{Y}_t$ (see Equation (1)) is given by:

$$\phi_{\mathbf{Y}}(\mathbf{u},t)=\prod_{i=1}^{N}\phi_{X_1^{(i)}}\left(u_i\theta_i+\mathrm{i}\frac{1}{2}\sigma_i^2u_i^2,t\right)\phi_{Z_1}\left(\sum_{i=1}^{N}\alpha_i\left(u_i\theta_i+\mathrm{i}\frac{1}{2}\sigma_i^2u_i^2\right),t\right).\qquad(2)$$

Indeed, we have

$$\phi_{\mathbf{Y}}(\mathbf{u},t)=\mathbb{E}\left[\exp(\mathrm{i}\mathbf{u}'\mathbf{Y}_t)\right]=\mathbb{E}\left[\mathbb{E}\left[\prod_{i=1}^{N}\exp(\mathrm{i}u_iY_t^{(i)})|G_t^{(i)},i=1,\ldots,N\right]\right].$$

Moreover, since $\theta_i G_t^{(i)} + \sigma_i W_{G_t^{(i)}} | G_t^{(i)} \sim$ **Normal**$(\theta_i G_t^{(i)}, \sigma_i^2 G_t^{(i)})$ are independent, we have

$$\phi_{\mathbf{Y}}(\mathbf{u}, t) = \mathbb{E}\left[\prod_{i=1}^{N} \mathbb{E}\left[\exp(\mathrm{i} u_i Y_t^{(i)}) | G_t^{(i)}\right]\right] = \mathbb{E}\left[\prod_{i=1}^{n} \exp\left(\mathrm{i}\left(u_i\theta_i + \mathrm{i}\frac{1}{2}\sigma_i^2 u_i^2\right)\left(X_t^{(i)} + \alpha_i Z_t\right)\right)\right].$$

Given the independence of the $X_t^{(i)}$'s, $i = 1, \ldots, N$ and Z_t, we finally obtain Equation (2).

The marginal characteristic functions are directly obtained from (2):

$$\phi_{Y^{(i)}}(u, t) = \mathbb{E}\left[\exp(\mathrm{i} u Y_t^{(i)})\right] = \left(1 - \mathrm{i}\frac{u\theta_i + \mathrm{i}\frac{1}{2}\sigma_i^2 u^2}{b_i}\right)^{-a_i t} \left(1 - \mathrm{i}\frac{\alpha_i}{c_2}\left(u\theta_i + \mathrm{i}\frac{1}{2}\sigma_i^2 u^2\right)\right)^{-c_1 t}. \tag{3}$$

From the marginal characteristic function (3), it is clear that each process $Y^{(i)} = \{Y_t^{(i)}, t \geq 0\}, i = 1, \ldots, N$ is a Lévy process (although not necessarily VG) since the marginal characteristic function can be rewritten as $\phi_{Y^{(i)}}(u, t) = (\phi_{Y^{(i)}}(u, 1))^t$.

The linear correlation between the processes $Y_t^{(i)}$ and $Y_t^{(j)}$ is time independent:

$$\rho_{ij} = \frac{\mathrm{Cov}\left(Y_t^{(i)}, Y_t^{(j)}\right)}{\sqrt{\mathrm{Var}\left[Y_t^{(i)}\right] \mathrm{Var}\left[Y_t^{(j)}\right]}}, \tag{4}$$

where

$$\mathrm{Cov}\left(Y_t^{(i)}, Y_t^{(j)}\right) = \theta_i\theta_j\alpha_i\alpha_j\frac{c_1}{c_2^2}t \quad \text{and} \quad \mathrm{Var}\left[Y_t^{(i)}\right] = \left(\theta_i^2\left(\frac{a_i}{b_i^2} + \alpha_i^2\frac{c_1}{c_2^2}\right) + \sigma_i^2\left(\frac{a_i}{b_i} + \alpha_i\frac{c_1}{c_2}\right)\right)t.$$

The parameter set of the generalized αVG model is $\{\theta_i; \sigma_i; \alpha_i; a_i; b_i, i = 1, \ldots, N; c_1, c_2\}$ leading to a number of parameters amounting to $5N + 2$. However, we can scale the parameter c_2 to 1 since multiplying c_2 by a constant c is equivalent to dividing the parameters α_i's by c. Moreover, for the sake of coherence, we will impose that the business time $G_t^{(i)}$ increases on average as the real time t, i.e. we impose that $\mathbb{E}\left[G_t^{(i)}\right] = \left(\frac{a_i}{b_i} + \alpha_i\frac{c_1}{c_2}\right)t = t$ which is equivalent to

$$\frac{a_i}{b_i} = 1 - \alpha_i\frac{c_1}{c_2}. \tag{5}$$

Hence the number of independent parameters is reduced to $4N+1$: $\{\theta_i; \sigma_i; \alpha_i; b_i, i = 1, \ldots, N; c_1\}$. We note that Equation (5) implies the following constraints on the model parameters

$$b_i\left(1 - \alpha_i\frac{c_1}{c_2}\right) > 0, \quad i = 1, \ldots, N \tag{6}$$

to ensure the positivity of the parameters a_i's. If we do not impose any other restrictions, the marginal characteristic functions (3) depend on all the model parameters which makes impossible the decoupling of the univariate implied volatility surface calibration and the correlation calibration. Indeed, once the calibration of the option surfaces is performed, there is no parameter left to calibrate the dependence structure. Hence, we can either use only univariate derivatives in the calibration procedure or take into account a penalty in the option surface calibration which measures the correlation goodness of fit. However, some additional conditions can be imposed to make the marginal characteristic functions independent on the model parameter c_1. This will lead to the original αVG model proposed by Semeraro (2008).

4. THE ORIGINAL αVG MODEL

The αVG model proposed by Luciano and Semeraro (Luciano and Semeraro (2010), Semeraro (2008)) is obtained by imposing the equality

$$b_i = \frac{c_2}{\alpha_i} \quad \forall i = 1, \ldots, N \tag{7}$$

such that the Gamma subordinator $G^{(i)}$ is Gamma distributed: $G_1^{(i)} \sim \mathbf{Gamma}(a_i + c_1, \frac{c_2}{\alpha_i})$. The condition (5) then becomes $a_i = \frac{c_2}{\alpha_i} - c_1$ and the marginal characteristic functions become independent on c_1:

$$\phi_{Y^{(i)}}(u,t) = \left(1 - \mathrm{i}\frac{\alpha_i}{c_2}\left(u\theta_i + \mathrm{i}\frac{1}{2}\sigma_i^2 u^2\right)\right)^{-\frac{c_2}{\alpha_i}t}.$$

The unitary time change associated to the ith underlying stock, $G^{(i)}$, is then $\mathbf{Gamma}(c_2/\alpha_i, c_2/\alpha_i)$ distributed and the ith asset log-return follows a $\mathbf{VG}(\sigma_i, \alpha_i/c_2, \theta_i)$ process. The number of free parameters amounts then to $3N + 1$ ($\{\theta_i; \sigma_i; \alpha_i, i = 1, \ldots, N; c_1\}$).

Under the reduced setting, the linear correlation between the asset-log returns can be rewritten as:

$$\rho_{ij} = \frac{\theta_i\theta_j\alpha_i\alpha_j}{\sqrt{\left(\frac{\theta_i^2}{b_i} + \sigma_i^2\right)\left(\frac{\theta_j^2}{b_j} + \sigma_j^2\right)}} c_1 \propto c_1. \tag{8}$$

5. CALIBRATION PROCEDURE

For the calibration of the original αVG model, we follow the same procedure as in Leoni and Schoutens (2008) and Luciano and Semeraro (2010) since we can then dissociate the calibration of the univariate option surfaces and the calibration of the correlations. On the other hand, the generalized αVG model can not be calibrated by following this methodology since the marginal characteristic functions depend on the whole parameter set. Hence, we can either perform the calibration of the option surfaces and the correlations simultaneously or calibrate the whole parameter set on univariate derivatives only.

5.1. The decoupling calibration

The decoupling calibration procedure proposed by Leoni and Schoutens (2008) might be applied for any multivariate model as long as the marginal characteristic functions are independent on at least one model parameter since the methodology consists of dissociating the univariate option surface calibration from the correlation calibration. Hence the calibration might be performed in two successive steps:

1. **calibration of the univariate option surfaces**

We first perform a simultaneous calibration of each option surface by using fast Fourier transform techniques such as the Carr-Madan formula (Carr and Madan (1998)). For a particular choice of the common parameters $\mathbf{p^c}$ (i.e. the parameters which are included in more than one marginal characteristic function), we calibrate the idiosyncratic parameters $\mathbf{p^i}$ (i.e. the parameters which only appear in one marginal characteristic function). We then repeat the procedure for a wide range of the common parameters. The optimal marginal parameter set $\mathbf{p^m} = \{\mathbf{p^c}, \mathbf{p^i}\}$ (i.e. the set of both the common and idiosyncratic parameters) is the parameter set which leads to the best fit of all the univariate option surfaces.

For the calibration of the marginal distributions, we consider a straightforward multidimensional extension of the widely used one dimensional root mean square error (RMSE) objective function by taking the mean of the marginal RMSE functionals:

$$\text{MRMSE} = \sum_{i=1}^{N} \frac{\text{RMSE}^{(i)}}{N} = \sum_{i=1}^{N} \frac{1}{N} \sqrt{\frac{\sum_{j=1}^{M^{(i)}} \left(P_j^{(i)} - \hat{P}_j^{(i)} \right)^2}{M^{(i)}}}, \tag{9}$$

where N is the number of underlying stocks, $M^{(i)}$ is the number of quoted options for the ith stock and $P_j^{(i)}$ and $\hat{P}_j^{(i)}$ denote the jth market and model option prices of the ith stock, respectively. The multivariate weighted RMSE objective function, MRMSE allows to calibrate separately each option surface. Indeed, we can minimize separately $\text{RMSE}^{(i)} = \text{RMSE}^{(i)}(\mathbf{p^i_i}|\mathbf{p^c})$, where $\mathbf{p^i_i} = \{\theta_i, \sigma_i, \alpha_i\}$ denotes the idiosyncratic parameter set of the ith underlying. Hence opting for the MRMSE objective function might turn out to significantly reduce the calibration time, especially for a large number of underlyings. In the particular case of the original αVG model we consider, the MRMSE actually reduces to N univariate VG calibrations since the marginal characteristic functions do not share any common parameter $\mathbf{p^c}$.

2. **calibration of the dependence structure**

   We fix the marginal parameters $\mathbf{p^m}$ to their optimal value according to the first step and we calibrate the correlation parameters $\mathbf{p^d}$ (i.e. the parameters which do not influence any marginal characteristic function, in the present case, $\mathbf{p^d} = c_1$) on the market implied correlations by minimizing a root mean squared objective function:

$$\text{RMSE}^{\rho} = \sqrt{\frac{1}{\frac{N^2-N}{2}} \sum_{i,j \neq i}^{N} \left(\rho_{ij} - \hat{\rho}_{ij}\right)^2} \tag{10}$$

   where ρ_{ij} and $\hat{\rho}_{ij}$ denote the market implied and the model correlations between the ith and jth asset log-returns, respectively. The model correlation $\hat{\rho}_{ij}$ is directly inferred by Equation (4).

5.2. The joint calibration

If no reliable estimate of the dependence structure can be inferred from liquid market quotes, we can then calibrate the whole parameter set of the generalized model on the univariate option surfaces only by following the procedure described in the option surface calibration phase of the decoupling calibration procedure. In other words, we can successively minimize $\text{MRMSE}|c_1 = \sum_{i=1}^{N} \frac{\text{RMSE}^{(i)}(\theta_i,\sigma_i,\alpha_i,b_i|c_1)}{N}$ and repeat the procedure for different values of the common parameter c_1.

On the other hand, a joint calibration procedure of the univariate option surfaces and the correlations is required when the marginal characteristic functions depend on the whole model parameter set if the correlation matching is a desired feature. It requires an adequate specification of the penalty function to take into account the correlation matching in the calibration procedure of the option surfaces. We propose to minimize the following objective function:

$$\text{MRMSEJ} = \sum_{i=1}^{N} \frac{\text{RMSE}^{(i)}}{N} + \alpha^{\rho}\text{MRMSE}^{*}\sqrt{\frac{1}{\frac{N^2-N}{2}} \sum_{j,k\neq j}^{N} (\rho_{jk} - \hat{\rho}_{jk})^2}, \tag{11}$$

where ρ_{jk} and $\hat{\rho}_{jk}$ denote the market implied and the model correlations between the jth and kth log-returns, respectively and where MRMSE^{*} is the optimal value of the multivariate root mean square error obtained by fitting the option surfaces only. The scaling of the correlation goodness of fit by this factor ensures that both terms of Equation (11) are of the same magnitude order. The parameter $\alpha^{\rho} \geq 0$ allows the user to specify the relative importance of the correlation matching; a parameter α^{ρ} equal to 0 indicating that the correlation calibration is not a desired feature and that the model is calibrated on the univariate option surfaces only.

6. CALIBRATION PERFORMANCE

The calibration of the original and generalized αVG models is performed for a time period ranging from the 2nd of June 2008 until the 30th of October 2009 with weekly quotes and therefore including the recent credit crunch. We consider a basket composed of four major stocks included in the S&P500 index, namely Apple, Exxon, Microsoft and Intl. Moreover, we infer the dependence structure of the asset log-returns from the CBOE S&P 500 implied correlation index which measures the expected average correlation between the index components (CBOE (2009)). The original model is calibrated by performing the decoupling calibration procedure described in Section 5.1 whereas the generalized model is calibrated on the univariate option surfaces only or by including a penalty term which assesses the correlation goodness of fit (referred to as *step 2*) (see Section 5.2).

6.1. The option surface goodness of fit

The MRMSE (9) which assesses the univariate option surfaces goodness of fit as well as the VIX volatility index which measures the future expected market volatility over the next 30 calendar days are shown on Figure 1. We observe that the Lévy models lead to a better fit of the univariate option surfaces than the Black-Scholes model except during the panic wave period which is characterized by a high value of the VIX and which occurred in the aftermaths of the bankruptcy of *Lehman*

Brothers, namely from October 2008 until December 2008. When calibrated on univariate option surfaces only, the generalized model is characterized by a slightly lower MRMSE than the original model. Moreover, taking into account the correlation goodness of fit in the calibration of the generalized model leads to an option surface fit of roughly the same quality as the original and the generalized model when this is calibrated on option surfaces only.

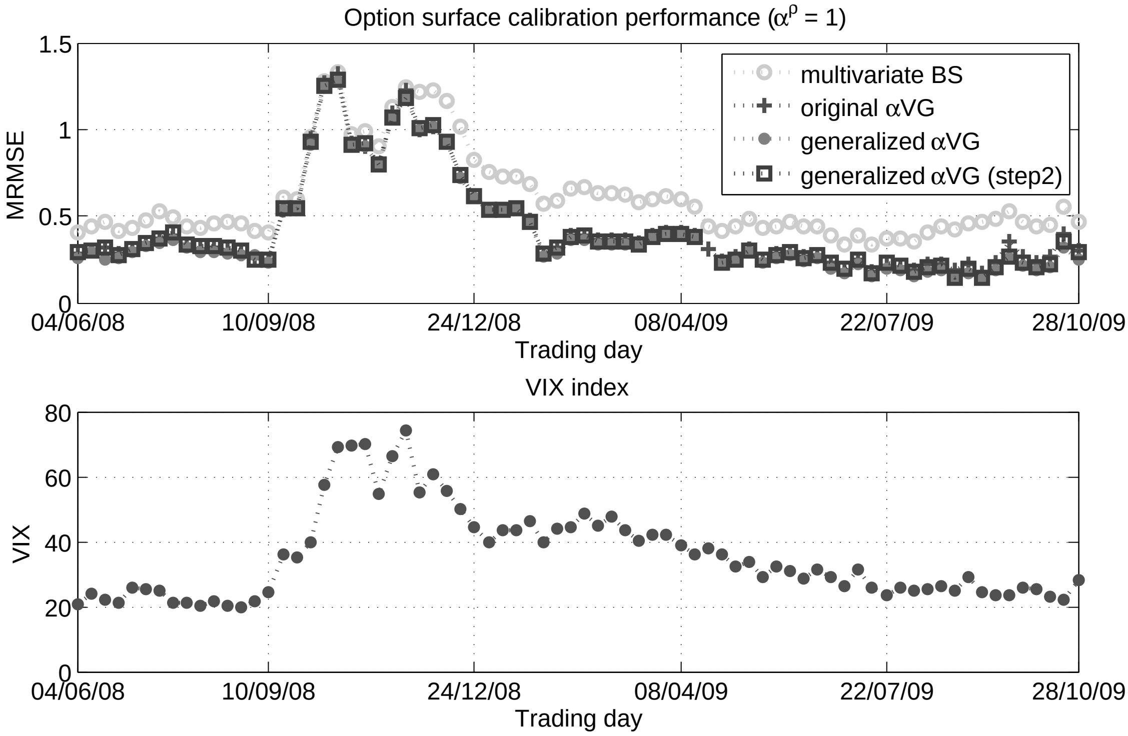

Figure 1: Evolution of the global option surface calibration performance (upper) and evolution of the VIX volatility index (lower) through time.

6.2. The correlation goodness of fit

Figure 2 shows the correlation RMSE (10) under the original and generalized αVG models. We clearly see that although the original αVG model has a free parameter to calibrate the linear dependence between the underlying stocks, i.e. c_1, it is usually not able to fit accurately the correlation structure. This gives some evidence against the use of the decoupling procedure to calibrate the original model and might be explained by two reasons: first there exists only one single parameter to fit the $\frac{N^2-N}{2}$ linear correlations between the N underlyings and secondly, imposing the constraint (5) that on average the business clock grows as the real time, implies some additional constraints on the subordinator parameters. Indeed, to ensure the positivity of the idiosyncratic subordinator parameters a_i's, we have to impose the conditions $c_1 < \frac{1}{\alpha_i}\ \forall i = 1, \ldots, N$; which is equivalent to impose an upper bound for c_1: $c_1 \in (0, \frac{1}{\max \alpha_i})$. Hence the range of admissible values of c_1 might be really small since the parameters α_i's are calibrated during the option surface calibration phase and can not be adjusted during the correlation calibration phase. Since under the original setting the correlation is proportional to c_1 (see Equation (8)), this might in turn severely restrict the range of attainable correlations. As it can be seen from Figure 3, the common subordinator parameter c_1 is usually set at the upper bound $\frac{1}{\max \alpha_i}$, which explained the poor fit of the dependence structure under the original αVG model.

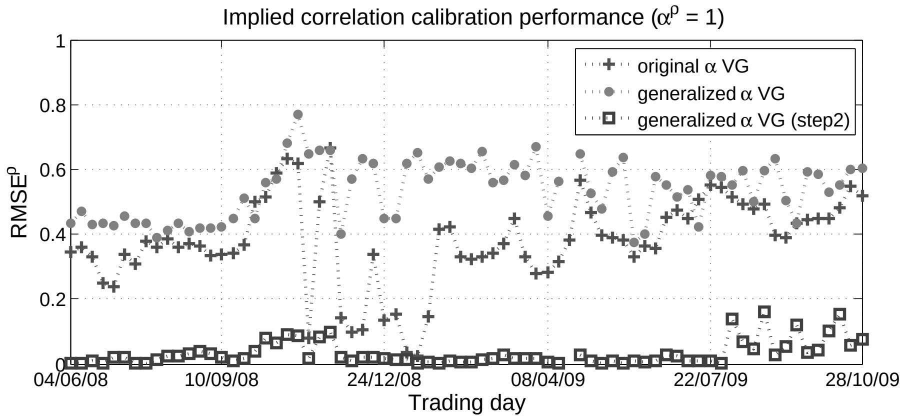

Figure 2: Evolution of the correlation calibration performance of the original and generalized models through time.

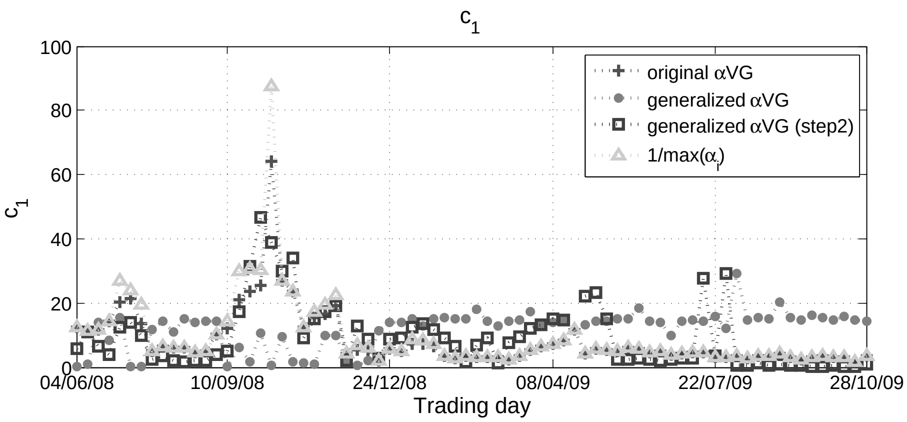

Figure 3: Evolution of the common subordinator parameter c_1 of the original and generalized models through time.

6.3. Influence of α^ρ

Figure 4 shows the influence of the parameter α^ρ on the option surfaces and the correlations goodness of fit for the trading day which leads to the highest value of RMSE$^\rho$ under the generalized

αVG model for $\alpha^{\rho} = 1$. We observe that it might be judicious to allocate more weight to the correlation goodness of fit in order to improve the correlation fit when the option surfaces RMSE is pretty low.

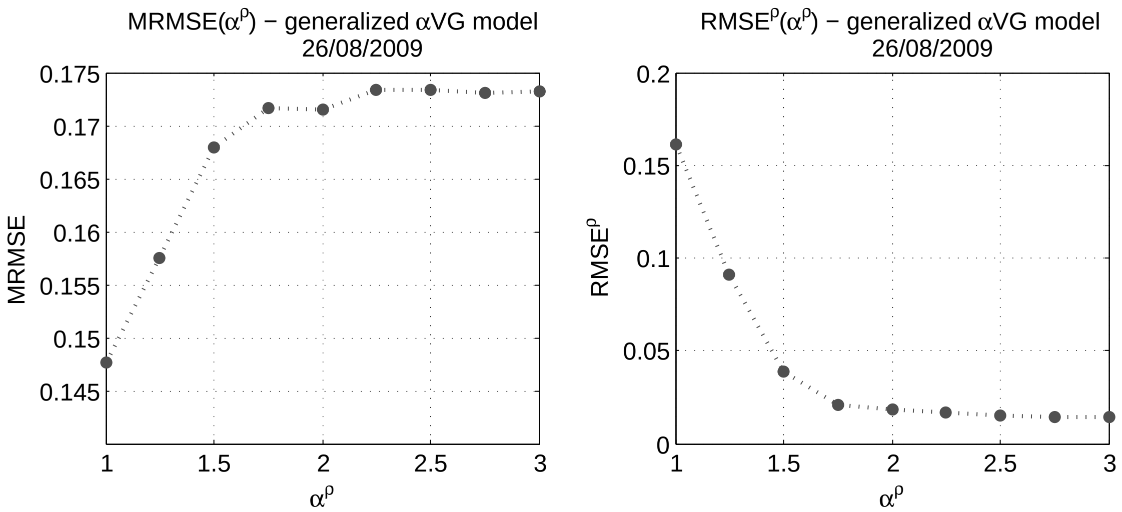

Figure 4: Influence of α^{ρ} for the generalized αVG model.

7. CONCLUSION

This paper features an extension of the αVG model, where the constraints on the Gamma subordinator parameters are relaxed. The such obtained generalized αVG model leads to marginal characteristic functions which remain of Lévy type but become dependent on the whole parameter set, which might be of a particular interest for practitioners as regards two criteria. First, the market-implied calibration does not require anymore the existence of a liquid market for multivariate derivatives which is nowadays pretty rare and secondly, the volatility, and hence the trading activity becomes a function of both the idiosyncratic and common subordinator settings, which is in line with the empirical evidence of the presence of both an individual and common business clock. The calibration of the two models has emphasized the fact that the correlation goodness of fit is significantly improved by performing a second calibration of the generalized model which takes into account a penalty term assessing the correlation goodness of fit into the option surface calibration optimizer. This paper also points out the shortfall of the decoupling calibration procedure in the case of the original αVG model. Indeed, imposing the condition that the business time grows on average as the calendar time implies an upper bound on the common parameter c_1 which is a function of the α_i's. By first calibrating the marginal parameters (including the α_i's) on the univariate option surfaces, we then limit severely the admissible value range of the parameter c_1 and consequently the value range of attainable correlations. In particular, the numerical study clearly shows that c_1 is usually set at its upper bound, giving some evidence against the use of the decoupling calibration procedure.

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QUANTIFICATION OF LIQUIDITY RISK IN A TWO-PERIOD MODEL

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Abstract

We present a static bank-run model for liquidity risk where a financial institution finances its risky assets by a mixture of short- and long-term debt. Insolvency can happen at any time until maturity. Short-term creditors have the possibility not to renew their funding at a fixed rollover date. We compute both insolvency and illiquidity default probabilities in a continuous time asset value model. Our implications show, in particular, that illiquidity risk is increasing in volatility and in the outside option ratio.

1. INTRODUCTION

Insolvency risk is defined as the risk that some obligors may default on their obligations or the risk of a deterioration in the credit quality of some investments resulting in unexpected losses. The credit crisis of 2007-2008 has dramatically shown that credit risk cannot only be reduced to insolvency risk but is also intertwined with liquidity aspects. The failures of Bear Stearns and Lehman Brothers are just two examples of bankruptcies due to a run by short-term creditors. Both institutions had capital cushions well above the Basel II minimal capital requirements, but had financed their long duration risky assets mostly through short-term debt. Thereby, they were heavily exposed to liquidity risk. It is now understood that short duration financing, for example through commercial papers and repo transactions, increases the exposure to panic runs which was one of the main causes of the credit crisis of 2007-2008. There already exists extensive theoretical literature on potential causes for bank runs due to illiquidity risk. The models of Bryant (1980), Diamond and Dybvig (1983) and Rochet and Vives (2004), for example, provide evidence for the fact that runs can occur due to self-fulfillment of depositor's expectations concerning the behavior of other depositors. Thus bank runs are a result of coordination problems among short duration depositors' roll-over decisions. He and Xiong (2009) extend these models of static coordination problems to a dynamic one where the firm's debt expirations follow a Poisson distribution with infinite time horizon. Thus, the decision of a short-term creditor whether to roll-over his debt or not after expiration depends also on his expectation about the rollover decisions of creditors maturing at different times.

Using a global game model Morris and Shin (2009) succeed in decomposing a financial institution's total credit risk in an insolvency risk and an illiquidity risk component. They propose a two-period model where short-term creditors face only one rollover decision at an interim time point. In particular, the authors analyze the policy implications on the balance sheet induced by the increase in total credit risk which arises from the additional illiquidity risk component. Moreover and in contrast to the aforementioned literature, they directly model the influence of future insolvency risk on the roll-over decision of short-term creditors and thus on illiquidity risk.

Inspired by Morris and Shin (2009) we construct in this paper a continuous time model with a mixture of short-term and long-term debt. More specifically, we consider a financial institution financing its risky assets using short- and long-term debt. Short-term debt earns lower return, but short-term creditors have the choice not to renew their funding at a fixed rollover date t^*. When rolling over their funding at date t^*, short-term creditors earn a return rate of r_S at final maturity T from the financial institution. When choosing not to renew their funding at date $t^*,$ they can earn a return rate of r^* on the market. The decision of a short-term creditor whether to roll over or not at date t^* surely depends on the outside return rate r^*. In case several short-term creditors choose not to roll over their funding, the financial institution might default due to illiquidity caused by a run of short-term creditors. We define illiquidity risk as the risk of a default due to a run by short-term creditors when the firm would otherwise have been solvent. The default probability due to illiquidity will then be specified by an illiquidity barrier such that when the asset value at the rollover date falls below this barrier a successful run will occur. We implement our model in a binomial tree framework. Our results show that illiquidity risk is increasing in volatility σ and in the ratio of the outside return over the return for short-term debt. These results are in accordance with previously derived implications by Morris and Shin (2009) for the situation of a discrete asset value model with a single rollover date. Moreover, we can explicitly quantify the increase in total default probability that is due to illiquidity risk.

2. FINANCING STRUCTURE

Suppose a financial institution finances a risky asset by short- and long-term debt. We model the value process $(V_t)_{t\geq 0}$ of the risky asset by a geometric Brownian motion

$$\frac{dV_t}{V_t} = \mu dt + \sigma dW_t$$

with constant drift μ and volatility $\sigma > 0$ where W is a standard Brownian motion.

Long-term debt with principle value of L_0 and maturity T is issued at time $t_0 = 0$. The promised (continuously compounded) rate of return for long-term debt is r_L per annum. So if there is no default, the value of long-term debt at maturity T is

$$L_T = e^{r_L T} L_0.$$

At initiation time $t_0 = 0$ short-term debt with principle value S_0 and maturity t^* is issued. Assume that at time t^* short-term creditors can decide whether they want to renew their funding or not. If some short-term creditors decide not to renew their funding, the financial institution will sell the corresponding short-term bonds to new creditors, if it does not default due to illiquidity at that time point.[1] Note that by this assumption the face value of short term debt at final maturity T is known in advance. If short-term debt is rolled over at time t^*, i.e. if the face value S_{t^*} is invested anew until time T and if the (continuously compounded) rate of return earned on short-term debt is r_S per annum which is assumed to be constant in time, then the face value at time T of short-term debt is

$$S_T = e^{r_S T} S_0.$$

The decision at time t^* of short-term creditors to roll-over or not depends on the return they can earn on the outside market as well as on the default probability of the financial institution in the time period $[t^*, T]$. We assume the (continuously compounded) outside rate of return to be constant and equal to r^* for all time periods. It can be set to equal the risk-free rate, however, it can also be the return from a risky project with a different financial institution. Then it should also incorporate the default probability of that project. It is a variable we will assume to be known and given in the market.

We assume that the financial institution also holds a cash amount M on the asset side which will be continuously compounded at the risk-free rate r.

3. DEFAULT PROBABILITIES

The current financial crisis has shown that many financial institutions have gone bankrupt even though their asset value was still greater than their debt value. In the above framework we calculate the default probability $\mathrm{PD}_{\mathbf{ins}}$ caused by insolvency of the financial institution and the default probability $\mathrm{PD}_{\mathbf{ill}}$ due to illiquidity at any time $t \in [0, T]$. The decomposition of total credit risk into these two components will allow us to hedge every risk component more effectively. Moreover, it will provide a method to determine an optimal composition of the liabilities side of the balance sheet to reduce illiquidity risk and thereby total credit risk of the financial institution. The key to calculate the default probability and its decomposition is to derive the default barriers due to insolvency and illiquidity.

In the computation of the default barriers due to insolvency and due to illiquidity of the financial institution we are motivated by the idea in Morris and Shin (2009). At the rollover date t^* the short-term creditors face a binary decision problem. They have to decide whether they rollover the debt or not depending on the corresponding returns from both decisions.

Unlike in Morris and Shin (2009) our model can accommodate a continuous time asset value model where insolvency can happen at any time until final maturity. Moreover, in Morris and

[1] If not all short-term creditors decide to run away, the financial institution should always be able to find some new creditors in the market.

Shin (2009) it is assumed that, when short-term creditors choose not to roll over their debt, they can always get the face value of their debt back and go to market to earn the return r^*. This, however, is only true when the financial institution is healthy and has enough cash to pay back the creditors. This assumption increases the incentive of the short-term creditors to run their debt. In our paper we model the return from not rolling over the debt according to the financial institution's condition. If it is healthy, the creditors will get the face value of their debt back; if it is in distress (with significant hair-cut of the asset value), then the creditors may get almost nothing.

Assume that the firm defaults due to insolvency at the first-passage time

$$\tau := \inf\{t \geq 0 : V_t \leq \alpha_t\},$$

where the *insolvency barrier* α_t is similar to Black and Cox (1976)

$$\alpha_t = \left(S_0 e^{r_S t} + L_0 e^{r_L t} - M e^{rt}\right) \cdot \rho$$

with $\rho \in [0, 1]$ being a safety covenant that determines how much of the firm value is available to compensate creditors and equity holders according to a pre-described seniority when the firm bankrupts. The default probability due to insolvency conditional on the information available at any time t_i is then

$$\begin{aligned} \mathrm{PD}_{\mathbf{ins}}(t^*) &= \mathbb{E}[1_{\{t^* \leq \tau \leq T\}} | V_{t^*}] \\ &= \mathbb{P}\left(\inf_{t^* \leq s \leq T} (V_s - \alpha_s) \leq 0 \middle| V_{t^*} \right). \end{aligned} \tag{1}$$

Besides the insolvency risk the firm might fail because of illiquidity. To compute the corresponding illiquidity barrier we need to fix some assumptions.

Assumption 1 *(a) A run can only happen at the decision time t^* and short-term debt can only be rolled over until final maturity T.*

(b) Assume that each short-term creditor believes that the firm will survive a bank run with a probability

$$\lambda(V_{t^*}) = \min\left\{1, \frac{\psi V_{t^*} + e^{rt^*} M}{S_{t^*}}\right\}$$

where ψ is the haircut rate.

(c) The short-term return rate r_S is strictly larger than the outside return rate r^*.

Assumption 1 (a) specifies our bank-run setting. Assumption 1 (b) describes the survival probability from a bank-run. Here the haircut takes values between 0 and 1. Intuitively, the ratio between the raised funds and the principle of short-term debt $\frac{\psi V_{t^*} + e^{rt^*} M}{S_{t^*}}$ represents the likelihood that the short-term creditors get back their face value of the debt. Because the creditors at most get their debt back, the above ratio is cut off at 1. The higher $\lambda(V_{t^*})$ is, the more funds the firm can raise, the more likely the short-creditors get their debt back. Assumption 1 (c) is necessary as short-term creditors would otherwise directly choose the risk-free outside option which would be more attractive.

To compute the illiquidity probability we first compute an illiquidity barrier and then analyze the event when the asset value at the rollover date t^* falls below this barrier. Denote the expected outside return rate the short-term creditor earns by investing in the market if he decides not to rollover debt at time t^* by $R^*(V_{t^*})$. Recall the market rate is denoted by r^*. The real market return is given by the market return times the survival probability from a bank run

$$e^{R^*(V_{t^*})\,(T-t^*)} = e^{r^*(T-t^*)} \cdot \lambda(V_{t^*}) \tag{2}$$

as the short-term debt is payed back in full only if the firm survives from a bank-run otherwise only a ratio of the face value will be payed back to short-term creditors. On the other hand, the short-term creditor earns r_S if he rolls over the debt provided that the firm does not default due to a bank-run at time t^* or due to insolvency in the final time period $[t^*, T]$. The expected return rate $R_S(V_{t^*})$ is given by the short-term debt return times the survival probability from insolvency for time $[t^*, T]$ multiplied by the survival probability from a bank-run at time t^*

$$e^{R_S(V_{t^*})\,(T-t^*)} = e^{r_S(T-t^*)} \cdot \mathbb{P}\left(\inf_{t^* \le s \le T} (V_s - \alpha_s) \ge 0 \middle| V_{t^*} \right) \cdot \lambda(V_{t^*}) \tag{3}$$

A run at time t^* occurs if the expected return rate $R_S(V_{t^*})$ is smaller than the expected outside return rate $R^*(V_{t^*})$. This provides an *illiquidity barrier* β_{t^*} at time t^* for the asset return V_{t^*} given as the solution of the following equation

$$e^{r^*(T-t^*)} = e^{r_S(T-t^*)} \cdot \mathbb{P}\left(\inf_{t^* \le s \le T} (V_s - \alpha_s) \ge 0 \middle| V_{t^*} \right) \tag{4}$$

Note that the survival probability from a bank-run at time t^* drops out of the equation for the illiquidity barrier. This is due to the fact that in case of a run by short-term creditors at time t^* they will get their debt back with the same probability $\lambda(V_{t^*})$ whether they roll over or not. Hence the decision of each short-term creditor whether to roll over or not actually does not depend on his believes about the behavior of other short-term creditors. It is only influenced by the insolvency probability of the financial institution.

For the computation of the *ex ante* default probability due to illiquidity, suppose we have already computed the default barrier β_{t^*} for the rollover date t^*. The financial institution can default because of a run at the rollover date t^* and because of insolvency at maturity T. At time 0 the survival probability that the financial institution will stay alive from 0 to T is

$$\mathbb{E}\left[\mathbb{1}_{\{V_{t^*} \ge \beta_{t^*}\}} \cdot \mathbb{1}_{\{\inf\limits_{0 < s \le T} \{V_s - \alpha_s\} \ge 0\}} \right]$$

From this, we can easily calculate the *ex ante* default probability for the period from 0 to T as

$$\mathrm{PD}_{\mathbf{total}}(t_0) = 1 - \mathbb{E}\left[\mathbb{1}_{\{V_{t^*} \ge \beta_{t^*}\}} \cdot \mathbb{1}_{\{\inf\limits_{0 < s \le T} \{V_s - \alpha_s\} \ge 0\}} \right] \tag{5}$$

which accounts for all bankruptcy scenarios such as the default at t^* because of a run by short-term creditors and also for the default because of insolvency in $[0, T]$ although the financial institution survives the rollover date.

We can then derive the default probability due to illiquidity as the difference between the total PD and the insolvency PD, i.e.

$$\mathrm{PD}_{\mathbf{ill}}(t_0) = \mathrm{PD}_{\mathbf{total}}(t_0) - \mathrm{PD}_{\mathbf{ins}}(t_0) \tag{6}$$

where $\mathrm{PD}_{\mathbf{ins}}(t_0)$ is computed using equation (1).

4. STATIC ANALYSIS OF DEFAULT PROBABILITIES

We have implemented our model in a binomial tree setting. Therefore, we assume the roll-over time to be at the midpoint $T/2$ to final maturity. To increase accuracy of the approximation of the continuous time asset value process in the tree, we introduced some interim time steps between times 0 and t^* and between t^* and T. We choose the time steps of the binomial tree to be equidistantly distributed with step size Δt such that $m\Delta t = T/2$ for some natural number $m > 0$, i.e. the binomial tree is of size $2m$.
As mentioned before, the asset value process is assumed to follow a geometric Brownian motion

$$V_t = V_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t\right). \tag{7}$$

A time discrete version of this process can be represented in a binomial tree if we set $u = e^{\sigma\sqrt{\Delta t}}$ and $d = \frac{1}{u}$ (see e.g. Hull (2010)). At each node in the binomial tree the asset value goes up with a probability

$$p = \frac{e^{\mu\Delta t} - u}{u - d}$$

and down with $1 - p$.

For the numerical results we chose the following set of parameters. The initial asset value $V_0 = 100$ and final maturity is set to $T = 1$ year. Drift and volatility of the risky asset equal $\mu = 6\%$ and $\sigma = 15\%$ resp.. The risk-free rate is set to $r = 1\%$. We assume a haircut value of $\psi = 70\%$ and a safety covenant of $\rho = 70\%$. Moreover, we assume a face-value of short-term debt of $S_T = 40$ and for long-term debt $L_T = 60$. For simplicity we assume the cash amount to be $M = 0$. Assume that the outside return r^* equals the risk-free rate of 1% while the promised return for short-term debt is $r_S = 4\%$. The difference between outside return r^* and return rate r_S corresponds approximately to the spread one currently obtains for an A-rating compared to risk-free. We assume a return rate for long-term debt of $r_L = 6\%$. Our numerical results are based on a binomial tree implementation with $m = 1000$ interim dates to increase accuracy of our calculations.

Figure 1 shows the decomposition of the total default probability into its insolvency and illiquidity components for increasing initial asset value V_0. For very low V_0 the financial institution will almost surely default due to insolvency, i.e. $\mathrm{PD}_{\mathbf{ins}} = 1$. In these cases the reason for a default is

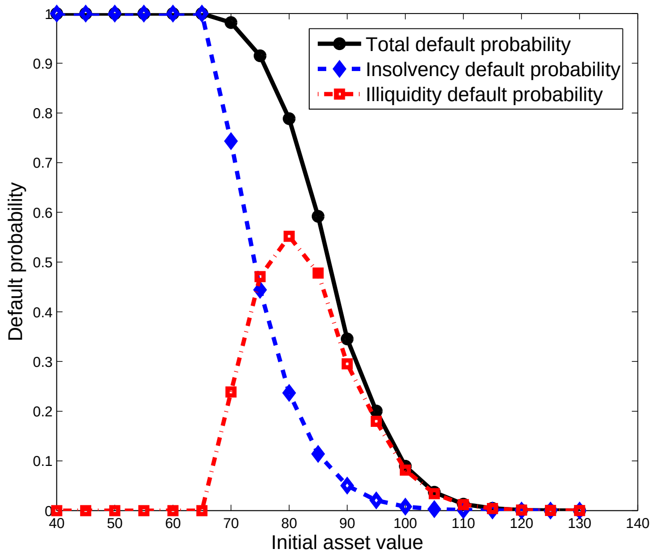

Figure 1: Influence of initial asset value on default probability

clearly insolvency and not any liquidity problems. Thus $\mathrm{PD}_{\mathbf{ill}} = 0$ in these situations. For higher initial asset values a default due to insolvency becomes more and more unlikely while the probability that the financial institution will default due to a run by short-term creditors increases up to a critical point. When the initial asset value is higher than some critical value, the illiquidity default probability decreases again as a run by short-term creditors becomes more and more unlikely. This is due to the fact that the probability that the asset value at the roll-over date is less than the illiquidity barrier becomes smaller and smaller. The figure shows that, when taking liquidity risk into account, the total default probability of the financial institution increases.

Moreover, in analogy to the results in Morris and Shin (2009) we obtain that illiquidity risk is increasing in volatility σ as is illustrated in Figure 2. This is also intuitive as higher volatility leads to higher fluctuations in the asset value and thus increases default risk in general. Thus all components of the total default probability, i.e. insolvency and illiquidity risk, increase with volatility σ.

Figure 3 illustrates the dependence of the illiquidity probability on the outside option return rate r^* and on the short-term debt return rate r_S. Similarly to the result of Morris and Shin (2009), we obtain that illiquidity risk is increasing in the outside option return rate r^* and correspondingly decreasing in the short-term debt rate r_S since the risk-free outside investment opportunity becomes more attractive.

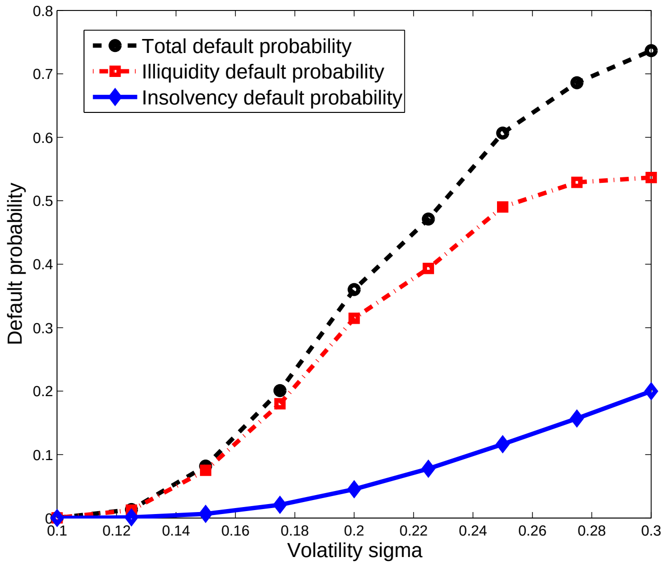

Figure 2: Influence of volatility σ on default probability

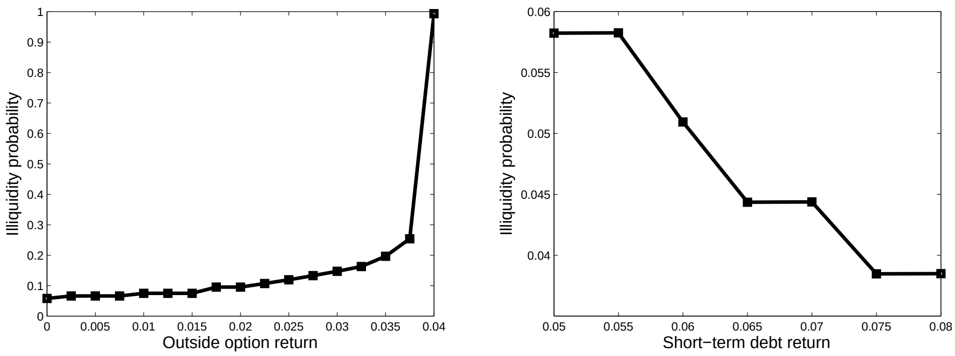

Figure 3: Influence of outside option return and short-term debt return on default probability

5. CONCLUSION AND DISCUSSION

In this paper we presented a model for the quantification of liquidity risk in a continuous time asset value framework where a financial institution finances its risky assets by both short- and long-term debt. Insolvency risk can occur at any time until maturity as in the Black and Cox (1976) framework. Short-term creditors have the possibility not to roll over their funding at a fixed decision time t^*. We succeeded in splitting total default probability into an insolvency and an illiquidity component and we studied their dependencies on the individual model parameters. Our implications show, in particular, that illiquidity risk is increasing in volatility σ and in the outside option return rate r^* and decreasing in the short-term return rate r_S. These results are in accordance with previously derived implications by Morris and Shin (2009) for the discrete asset value situation.

The extension to a multi-period setting where short-term creditors can decide whether to roll over their funding or not at a finite number of roll over dates is current work in progress. In such a setting the illiquidity barrier at early decision points will depend not only on the insolvency probability of the financial institution but also on the survival probability from a bank-run at later decision points. Thereby we obtain a dynamic coordination problem among short-term creditors rollover decisions. Studying the optimal debt structure with and without liquidity risk under some additional constraints in such a dynamic model setting is also current work in progress.

An interesting extension would be to consider the continuous bank-run case, i.e. where creditors can decide at any time to run the bank. We will investigate in future work the optimal time point τ for short-term creditors to run the bank. Short-term creditors can earn the return rate r_S until they decide to run the bank (as long as the bank is still solvent). After running the bank they earn the outside return rate r^* until maturity if there is no bank run at time τ. Note that on the event $\tau = T$ there is no real bank run, so the creditor will receive the principal value of his debt if the insolvency does not occur. The creditor will choose an optimal stopping time τ to maximize his expected return.
Since the outside return rate r^* is not equal to the short-term return rate r_S, we obtain a time inconsistent optimization problem, meaning that the creditors' preferences are changed over time. Recall, that e.g. for American options, the investors always earn the risk-free interest rate r no matter whether they continue or exercise their options. This is one of the key differences between our model and American option framework. Solving this time inconsistent optimization problem is current work in progress.

Acknowledgement

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DELTA AND GAMMA HEDGING OF MORTALITY AND INTEREST RATE RISK

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1. INTRODUCTION

This paper is based on Luciano et al. (2011) and studies the hedging problem of life insurance policies, when the mortality rate is stochastic. In recent years, the literature has focused on the stochastic modeling of mortality rates, in order to deal with unexpected changes in the longevity of the sample of policyholders of insurance companies. This kind of risk, due to the stochastic nature of death intensities, is referred to as systematic mortality risk. In the present paper we deal with this, as well as with two other sources of risk life policies are subject to: financial risk and non-systematic mortality risk. The former originates from the stochastic nature of interest rates. The latter is connected to the randomness in the occurrence of death in the sample of insured people and disappears in well diversified portfolios.
The problem of hedging life insurance liabilities in the presence of systematic mortality risk has attracted much attention in recent years. It has been addressed either via risk-minimizing and mean-variance indifference hedging strategies, or through the creation of mortality-linked derivatives and securitization. The first approach has been taken by Dahl and Møller (2006) and Barbarin (2008). The second approach was discussed by Dahl (2004) and Cairns et al. (2006b) and has witnessed a lively debate and a number of recent improvements, see f.i. Blake et al. (2010) and references therein.
We study Delta and Gamma hedging. This requires choosing a specific change of measure, but has two main advantages with respect to risk-minimizing and mean-variance indifference strategies. On the one side it represents systematic mortality risk in a very intuitive way, namely as the difference between the actual mortality intensity in the future and its “forecast” today. On the other side, Delta and Gamma hedging can easily be implemented and adapted to self-financing constraints. It indeed ends up in solving a linear system of equations. The comparison with securitization works

as follows. The Delta and Gamma hedging complements the securitization approach strongly supported by most academics and industry leaders in two senses. On the one hand, as is known, the change of measure issue on which hedging relies will not be such an issue any more once the insurance market, thanks to securitization and derivatives, becomes liquid. On the other hand, securitization aims at one-to-one hedging or replication, while we push hedging one step further, through local, but less costly, coverage.
The paper proceeds as follows: first we present the general framework for representing stochastic mortality through Cox processes, then we focus on two particular affine processes and we show they satisfy an HJM-condition for no arbitrage after an appropriate change of measure. Then we describe Delta and Gamma hedging of pure endowments and we provide an example calibrated on the UK market.
We refer the reader to Luciano et al. (2011) for details, proofs and a more comprehensive account of the technique we present here and its application to a UK calibrated example.

2. THE MODEL FOR MORTALITY AND FINANCIAL RISK

Following a well established stream of actuarial literature, we adopt the setting of risk-neutral interest rate modelling to represent stochastic mortality. Hence, we represent death arrival as the first jump time of a doubly stochastic process. To enhance analytical tractability, we assume a pure diffusion of the affine type for the spot mortality intensity. Namely, the process has linear affine drift and instantaneous variance-covariance matrix linear in the intensity itself.
In particular, we consider a probability space $(\Omega, \mathbb{F}, \mathbb{P})$ and restrict our attention to two affine processes, belonging to the Ornstein-Uhlenbeck and the Feller class, for mortality intensity λ:

- Ornstein-Uhlenbeck (OU) process without mean reversion:

$$d\lambda_x(t) = a\lambda_x(t)dt + \sigma dW_x(t)$$

- Feller Process (FEL) without mean reversion:

$$d\lambda_x(t) = a\lambda_x(t)dt + \sigma\sqrt{\lambda_x(t)}dW_x(t)$$

with $a > 0$, $\sigma \geq 0$, and W_x a univariate Brownian motion under $\mathbb{P}$.

These processes turn out to be appropriate choices for the description of human mortality, as already pointed out by Luciano and Vigna (2008) and have already been used in the modelling of dependent lives (see Luciano et al. (2008)).
We then recall the definition of forward mortality intensity, which we define as $f_x(t, T)$. We point out that the risk factor against which one could be interested to hedge its positions is the difference between the (stochastic) future realization of the mortality intensity at a future time t and the forward intensity, which can be interpreted as its "best forecast" today. We show that in the affine case forward intensities can be easily computed as affine functions of the solutions α and β of the Riccati ODEs associated to the intensity process:

$$f_x(t, T) = -\alpha'(T - t) - \beta'(T - t)\lambda_x(t) = -\alpha'(T - t) - \beta'(T - t)f_x(t, t).$$

For what concerns interest-rates, we model the instantaneous forward rate directly as

$$dF(t,T) = A(t,T)dt + \Sigma(t,T)dW_F(t) \tag{1}$$

where the real functions $A(t,T)$ and $\Sigma(t,T)$ satisfy the usual assumptions for the existence of a strong solution to (1), and W_F is a univariate Brownian motion under $\mathbb{P}$ independent of W_x for all x.

3. CHANGE OF MEASURE AND HJM RESTRICTION ON FORWARD DEATH INTENSITIES

After having defined both markets, we tackle the issue of finding an appropriate change of measure. Following Dahl and Møller (2006) among the possible changes, we select the minimal one, the one which permits to remain in the Ornstein-Uhlenbeck and Feller class. We follow a common assumption and set the premium on non-systematic mortality risk to zero, which is equivalent to assuming that the portfolio of insured people is well diversified. We further parametrize the measure by assuming that the premium for systematic mortality risk is constant and that the interest-rate market is complete. Hence, under these assumptions, the fair premium and the reserves of life insurance policies can be computed as expected values under the measure $\mathbb{Q}$ equivalent to $\mathbb{P}$.
We are interested in pure endowment contracts starting at time zero and paying one unit of account if the head x is alive at time T. The fair premium or price of such an insurance policy, $P(0,T)$, given the independence between the financial and the actuarial risk, is:

$$P(0,T) = S_x(T)B(0,T) = e^{\alpha(T)+\beta(T)\lambda_x(0)} E_{\mathbb{Q}} \left[-\exp\left(\int_0^T r(u)du \right) \right],$$

where $S_x(T)$ is the survival probability of the head x from time 0 to T, $B(0,T)$ is the price at zero of a zero-coupon bond with maturity T and $r(t)$ denotes the short rate at time t. The value of the same policy at any future date t is:

$$\begin{aligned} P(t,T) &= S_x(t,T)B(t,T) \\ &= E_{\mathbb{Q}} \left[\exp\left(-\int_t^T \lambda_x(s)ds \right) \mid \mathcal{G}_t \right] E_{\mathbb{Q}} \left[-\exp\left(\int_t^T r(u)du \right) \mid \mathcal{H}_t \right], \end{aligned}$$

where $\mathcal{G}_t$ is the sigma-algebra containing all the information on mortality and $\mathcal{H}_t$ contains all the information on the financial market up to time t (see Luciano et al. (2011) for details).
Hence, we can define a “term structure of pure endowment contracts”. The last expression, net price of the initial premium, is also the time t reserve for the policy, which the insurance company will be interested in hedging. Notice that we did not impose a no-arbitrage condition on the market for these instruments. Once the change of measure has been performed, we can write $P(t,T)$ in terms of the instantaneous forward intensity and interest rate (f and F respectively):

$$P(t,T) = \exp\left(-\int_t^T \left[f_x(t,u) + F(t,u) \right] du \right).$$

In this setting, Cairns et al. (2006a) point out that the HJM no-arbitrage condition typical for the financial market can be translated into an equivalent HJM-like condition for forward death intensities. Usually, the aspect of the HJM condition on the insurance market is imposed a priori.
In our paper we show that, for the two non-mean reverting processes for the mortality intensity we consider, namely OU and FEL, there exists an infinity of probability measures — equivalent to the historical one — in which forward death intensities satisfy an HJM condition. No-arbitrage holds under any of these measures, characterized by a constant risk-premium on mortality, even though it is not imposed a priori.

4. DELTA AND GAMMA HEDGING

After having selected the appropriate change of measure, we can avoid using risk minimizing or mean-variance indifference strategies. We can instead focus on Delta and Gamma hedging.
For the sake of simplicity we assume that the market of interest rate bonds is not only arbitrage-free but also complete. First, we consider a pure endowment hedge in the presence of systematic mortality risk only. Then, under independence of mortality and financial risks, we provide an extension of the hedging strategy to both these risks. We show that our technique simply involves the solution of linear systems of equations.
We show that when the mortality intensity follows the OU process, the reserve of the longevity bond $P(t, T)$ (i.e. the survival probability when the interest rate is deterministic and null) can be written as an easily tractable exponential affine function of the risk factor. Moreover, its change dP — through Ito's lemma — can be simply written as a function of its first and second order sensitivities to the risk factor, which — in the financial literature — are usually referred to as Delta and Gamma.
We then describe the Delta and Gamma coverage technique for pure endowments, using as hedging tools either pure endowments or zero-coupon survival bonds for mortality risk and zero-coupon-bonds for interest rate risk. Since all these assets can be understood as Arrow-Debreu securities — or building blocks — in the insurance and fixed income market, the Delta and Gamma hedge could be extended to more complex and realistic insurance and finance contracts.

5. APPLICATION

Finally, we provide a calibrated example. We use UK mortality rates for the male generation born in 1945 and we calibrate the Hull-White model for interest rates on the UK government bonds market. We compute the Delta and Gamma factors for different maturities and we provide the computation of the Delta and Gamma hedged strategies for an insurer who has issued a pure endowment on a certain head with maturity 15 or 30 years. We show strategies which involve both the use of longevity bonds/pure endowments only or also the use of interest-rate bonds and which can be self-financing or not.

Our application shows that:

1. the unhedged effect of a sudden change on mortality rate is remarkable, especially for long time horizons;
2. the corresponding Deltas and Gammas are quite different if one takes into consideration or ignores the stochastic nature of the death intensity;
3. the hedging strategies are easy to implement and customize to self-financing constraints;
4. Delta and Gamma are bigger for mortality than for financial risk.

In particular, we find that the effects of comparable changes in the interest-rate and in the mortality rate lead to comparable effects on the prices of policies. This is a clear indication that hedging systematic mortality risk could be very important for a life-insurer.

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POSTER SESSION

SENSITIVITY ANALYSIS AND GLOBAL RATING FOR ABSs

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Asset backed securities (ABSs) are structured finance products backed by pools of assets and created through a securitization process. The ratings of asset backed securities are partly based on quantitative models for the defaults and prepayments of the assets in the pool. This quantitative assessment is based on assumptions and estimations of input parameters that are affected by uncertainty. The uncertainty in these variables propagates through the model and produces uncertainty in the ratings. We propose to work with global sensitivity analysis techniques to investigate ABS ratings sensitivity to the input parameters and we introduce a novel structured financial rating to take into account uncertainty in assessment.

1. INTRODUCTION

The rating and valuation of securitization transactions have been in focus the last years due to the enormous losses anticipated by investors and the huge amount of downgrades among structured finance products. A rating is an assessment of the different risks inherent in a structure and how well these risks are mitigated. The rating process is based on both a quantitative assessment and a qualitative analysis, which assess the originator's and the servicer's operations and legal issues concerning the transfer of the assets from the originator to the issuer. For the quantitative assessment, models with one or more parameters are used to generate defaults and prepayments in the asset pool. Typically the input parameters are unknown and estimated from historical data or given by expert opinions. In any way, the values used for the parameters are uncertain and these uncertainties are propagated through the model and generate uncertainty in the rating output (see Jönsson and Schoutens (2009), Jönsson and Schoutens (2010), and Jönsson et al. (2009)).

[1]The views expressed are the author's and do not necessarily represent the views of BNP Paribas Fortis Bank and BNP Paribas Group.

There have been an increased attention to the rating of asset backed securities due to the credit crisis of 2007 - 2008 (see Moody's Investor Service (2000) and Moody's Investor Service (2009)). The objectives of this paper are twofold. Firstly, we advocate the use of global sensitivity analysis (SA) techniques to enhance the understanding of the main sources of output uncertainties.We quantify the percentage of output variance that each input factor is accounting for and we also detect how interactions among input parameters affect the rating variability.

Secondly, we propose a novel rating approach called *global rating*, that takes this uncertainty in the output into account when assigning ratings to tranches. The global ratings should therefore become more stable and reduce the risk of cliff effects, that is, that a small change in one or several of the input assumptions generates a dramatic change of the rating. The global rating methodology proposed gives one answer of a way forward for the rating of structure finance products.

2. ASSET BACKED SECURITIES

Asset backed securities (ABSs) are securities created through a securitization process whose value and income payments are backed by a specific pool of underlying assets (see Fabozzi and Kothari (2008)). Illiquid assets cannot be sold individually so they are pooled together and transferred to a shell entity specially created to be bankruptcy remote (Special Purpose Vehicle or SPV) which in turn issues notes (liabilities) to investors with distinct risk return profiles and different maturities: senior, mezzanine and junior notes.

The assessment of the ABS is related with the risks inherent in the structure. The ratings are indicators of the credit risk embedded in these instruments. The assessment of a final rating for asset backed securities relies on modelling of the cashflows produced by the assets, the collections of these cashflows and the distribution of the cashflows to the liabilities according to a payment priority. The modelling of the cashflows from the asset pool is based on default and prepayment models of different level of sophistication. We focus just on the default models and the prepayments are not included in the analysis for simplicity.

By using Monte Carlo simulations, different default scenarios are generated by first sampling a cumulative portfolio default rate from a default distribution and then distributing this default rate over time using a default curve. The default distribution of the pool is assumed to follow a Normal Inverse distribution in accordance with Moody's methodology for granular portfolios and the default curve is modelled by the Logistic model. In the sequel, we will calculate the **Expected Loss** and **Expected Average Life** of the notes. Having estimated these two quantitative outputs, we can map them into a qualitative Moody's rating using Moody's Idealised Cumulative Expected Loss Table.

3. SENSITIVITY ANALYSIS

We fill the need of investigating the rating sensitivity with respect to input assumptions by using sophisticated methods. We have already seen that the assessment of this financial product is based on a quantitative model containing some input parameters whose values are affected by uncertainty. This uncertainty propagates through the model and generates uncertainty in the rating output. By using a sensitivity analysis, we want to investigate on this uncertainty. Different sensitivity analysis techniques can be followed to test the sensitivity of a model, ranging from the global variance method (see Saltelli (2002), Saltelli et al. (2008) and Saltelli et al. (2004)), which decomposes quantitatively the total output variance into contributions of each input, to the simplest class of the screening tests which provides a qualitative information by varying one factor at a time. The start point for both of them is to run the model different times in order to take into account that each input can assume a different value: from each parameter setting of the input factor, we evaluate the model. The first class requires a high number of model evaluations and an extreme computational cost but we take advantage of using it because we get the contribution of each input factor to the variance of the output taking into account the interactions among factor. Within the screening methods, the elementary effects method (EE method) identifies important factors with few simulations.

Because of the ABS structure's complexity, our model is computationally expensive and the EE method is very well suited to screen the input space in a first step. All the non-influential factors will be determined and their values will be fixed without affecting the output variance of interest. Following, the variance based method will be applied to quantify and distribute the uncertainty of our model among the parameters identified to be influential by the elementary effect.

4. UNCERTAINTY AND SENSITIVITY ANALYSIS RESULTS

The sensitivity analysis is performed on a structure where the collateral pool's characteristics, the structural characteristics and the waterfall have been fixed. Without loss of generality, the investor is assumed to be informed about them, so that these features do not affect the output variance of interest. Assuming the default distribution of the pool to follow a Normal Inverse distribution and the default curve to be modelled by the Logistic model, the uncertain input factors in the sensitivity analysis are related to the parameters of both distributions and also to the default timing and the recoveries. Each one of these inputs can assume a discrete number of values within a range of variation that have to be fixed at the beginning.

The fundamental qualitative output in our study is the rating of the ABSs, addressing the loss a note investor might suffer. Having a look at the empirical distribution of these ratings on each note, we obtain information on the uncertainty in the model. The analysis points out that the problem of providing a credible rating gets more difficult for the mezzanine tranche; the uncertainty is too wide and the possibility of failure in the rating determination is too high and must be reduced. The senior tranche instead looks to provide good and reliable results. The reasons of this good or bad performance are not explicit to us. It would be interesting to find out which uncertainties are driving these results. Under these circumstances it is wise to investigate through sensitivity analysis techniques which variable drives most of this uncertainty. We know that each rating has been derived from mapping the **Expected Average Life** and the **Expected Loss** of the notes, thus these two values are the quantitative outputs the sensitivity analysis should investigate in order to assess the influence of the unknown inputs in the ABS ratings.

The exploration of the input space by using the EE method (see bar plots of μ^* in Figure 1) leads to the conclusion that among all seven input factors just five of them (μ_{cd}, Coeff. Variation, RR, t_0, and c) play a major role in determining the uncertainty in the output rating. This leads to the need of including them in a more sophisticated analysis.

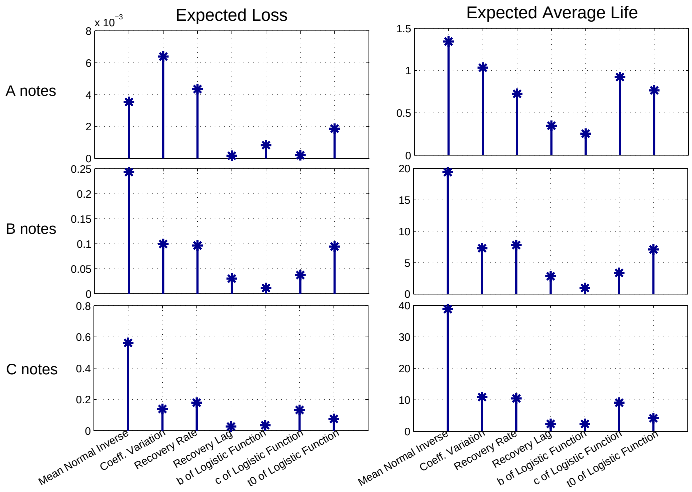

Figure 1: Bar plots of the μ^* values.

We therefore proceed to perform a quantitative sensitivity analysis in order to assess the importance of each factor by computing its contribution to the variability of the output. By using the variance based method we calculate the exact percentage of the output variance removed by learning the true value of an input factor taking into account the individual effect and the interactions in which each factor is involved (see Figure 2).

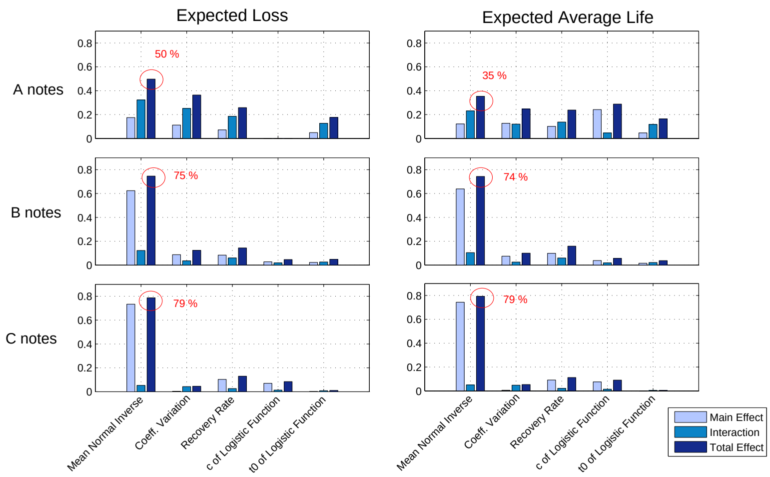

Figure 2: First Order and Second Order Sensitivity Index

The variance based method provides an encouraging insight: the mean cumulative default in the ABSs modeling is the main contributor to the uncertainty in the output. As this is a controllable factor we are encouraged to carry out further analysis searching for the optimal value for this factor thus reducing uncertainty in the analysis outcome. The mezzanine tranche that has been detected to be unreliable due to the uncertainty, can be controlled if we try to better assess the value for the mean cumulative default. If this would not be the case we would have accepted the fact that most of the uncertainty in the mezzanine tranche is due to an intrinsic problem and therefore unavoidable.

5. GLOBAL RATING

We have already seen that the uncertainty in the input parameters propagates through the model and generates uncertainty in the outputs. We propose to use a new strategy which takes into account this uncertainty when rating ABSs. We call this new approach a *global rating*. The global approach derives the rating of a note from the empirical distribution of ratings generated from the global scenarios. This new scale is superimposed on a rating scale used by a rating agency or by a financial institution and it is based on a percentile mapping of the underlying rating scale, that is, to assign a global rating to a tranche if a predetermined fraction of the ratings generated using the global scenarios is better than or equal to a given underlying rating. In order to take into account the uncertainty, rather than using a single rating that is very accurate but may easily change when changing one input value, we would prefer to use a global rating that incorporates several underlying ratings resulting to be more stable.

As can be seen in Table 1 the idea is to let the global rating reflect a range of possible credit risks. Hence, to set up the global rating scale we first have to decide on the ranges of the credit risk and of the underlying rating scale. Secondly, we have to choose the fraction of rating outcomes that should be laying in the credit risk range. As first attempt, we have defined the scale with respect to the 80th percentile of the local rating scale (in this case Moody's ratings) and we find the global rating to be A, D, and E for the senior, mezzanine and junior tranche respectively.

Global Rating	Credit Risk Range	Moody's
A	Low	A3–Aaa
B	Low to Medium	Baa3–Aaa
C	Low to High	Ba3–Aaa
D	Low to Higher	B3–Aaa
E	Low to Highest	N.R.–Aaa

Table 1: The global rating scale and the corresponding ranges in credit risk and in Moody's rating scale

6. CONCLUSION

The valuation of different types of asset backed securities (ABSs) have been in focus the last years due to the enormous losses anticipated by investors and the huge amount of downgrades among structured finance products. The assessment of the risk inherent in an ABS structure and how well these risks are mitigated is detected by the ratings.

By the uncertainty analysis, we figure out that the mezzanine tranche seems to be unreliable due to the uncertainty that is too wide so that the possibility of failure in the rating determination is too high. The senior tranche instead seems to provide good and reliable results. By using sensitivity analysis techniques we detect the main sources of uncertainty in the ratings of asset backed securities (ABSs) and we quantify the uncertainty in the model due to each different sources of uncertainty in the assessment. In particular, the mean cumulative default rate plays a major role in determining a rating in the senior, mezzanine and junior tranche. As our second research line, we introduce a methodology to evaluate the asset backed securities based on percentiles, that takes into account the uncertainty and produces more stable ratings. We propose to work with a new rating concept called global rating, where a new rating scale is used indicating the range of the credit risk of an asset backed security.

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A NOVEL BOOTSTRAP TECHNIQUE FOR ESTIMATING THE DISTRIBUTION OF OUTSTANDING CLAIMS RESERVES IN GENERAL INSURANCE

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I present a novel non-parametric bootstrap method for estimating claims reserves for claims triangles, which I call the *local chain ladder bootstrap* technique. The method is simple and can readily be implemented in a spreadsheet. The behaviour of the method is illustrated on a simulated claims triangle, in which the distribution of reserves is known. It appears that the distributional shape of the reserves is estimated quite well, however there is a location bias. If this bias could estimated and be corrected for, then the method might well be of interest to practitioners.

1. INTRODUCTION

For many years the chain-ladder technique for estimating claims reserves has been widely used (Taylor 2000). Two reasons can be put forward for this: (i) a chain-ladder analysis is simple for practitioners to implement; (ii) the results from a chain-ladder analysis usually accord reasonably well with the expectations of experienced practitioners. One restriction of the basic chain-ladder technique is that it only provides a point estimate of the reserves. In recent years there have been a number of proposals to overcome this limitation in order to model the distribution of reserves (for a good overview see Wüthrich and Merz (2008)). Several of these proposals use stochastic models that either are based upon the chain-ladder technique, or are constructed in order to reproduce the chain-ladder estimates in expectation. Interest in these models has grown among academic actuaries because of the realization that the variability in the reserves can be more informative than only a simple point estimate. These concerns have also prompted interest from practising actuaries, however the use of such models in practise appears limited. Two reasons could be put forward for why this is. Firstly, such models can be quite complex mathematically, and so be difficult to implement. Secondly, there is no agreement on which of the stochastic models is best to use. Even a single stochastic model may have several variants distinguished, for example, by choice of distributions (gamma, over-dispersed Poisson, normal, log-normal, etc.). In addition, some models based upon positive distributions might not be applicable to data in which incremental claims are negative in one or more development years.

This variety of stochastic models, and the lack of consensus on which model is appropriate for particular data (eg: why use a gamma distribution instead of a log-normal distribution?), means that a practising actuary could have difficulty in justifying the use of a particular model to a regulator. One approach would be to estimate the distribution of reserves using several models, and see if they approximately agree in their predictions: the one providing the most conservative estimates might then used. This would not be a problem if the models were simple to implement. However, as already mentioned earlier, some models are quite complex, and so such a strategy could be beyond the resources available to many practising actuaries.

Here I present a new method of estimating reserves based on a simple bootstrap simulation method. The result of the simulation is a sample from which the distribution of reserves may be constructed and analysed numerically. The method is non-parametric in nature — no distributional assumptions, for example, about individual claim sizes or their number are made. The model can cope with negative incremental claims.

The plan of this paper is as follows. The next section introduces some notation and summarises the standard chain-ladder technique. I then present the new bootstrap method, which I call the *local chain ladder bootstrap* model. Three variants of the method are presented and applied to a simulated triangle whose distribution of true completions are known.

2. THE CHAIN LADDER TECHNIQUE

It is assumed that the reader is familiar with the standard chain ladder technique: expositions may be found in Taylor (2000) and Wüthrich and Merz (2008). I shall work with claims triangles, in which rows label the period of origin of the claim. The columns represent the development year. The data in the upper triangle represents the amount paid out on claims. Inflation etc. is not modelled. Following (England and Verrall 2002), I use C_{ij} to represent the incremental claim amount in origin year (row) i and development year (column) j. With this notation, the claims are laid out as in Table 1.

	Development year						
Origin year	1	2	$\cdots$	j	$\cdots$	$n-1$	n
1	$C_{1,1}$	$C_{1,2}$	$\cdots$	$C_{1,j}$	$\cdots$	$C_{1,n-1}$	$C_{1,n}$
2	$C_{2,1}$	$C_{2,2}$	$\cdots$	$C_{2,j}$	$\cdots$	$C_{2,n-1}$	
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$		
i	$C_{i,1}$	$C_{i,2}$	$\cdots$				
$\cdots$	$\cdots$	$\cdots$	$\cdots$				
$n-1$	$C_{n-1,1}$	$C_{n-1,2}$					
n	$C_{n,1}$						

Table 1: Incremental claims-triangle format

Summing a row year up to a certain development period leads to *cumulative claims*,

$$D_{ij} = \sum_{k=1}^{j} C_{ik},$$

The chain ladder technique consists of using the values in the development triangle to construct so-called *development factors* λ_j, for each development year $j = 2, \ldots, n$,

$$\lambda_j = \frac{\sum_{i=1}^{n-j+1} D_{i,j}}{\sum_{i=1}^{n-j+1} D_{i,j-1}}.$$

These are used as multiplicative factors to fill-in the lower half of the cumulative claims triangle, in a recursive manner according to the formula:

$$\hat{D}_{i,j+1} = \lambda_j \hat{D}_{i,j}, \text{ for } j > n - i - 1$$

where we define the diagonal entries:

$$\hat{D}_{i,n-i+1} = D_{i,n-i+1} \text{ for } i = 2, \ldots, n.$$

The values in the final column of the completed square give the estimates of the total claims for each year — also known as the *ultimate claims*. Subtracting from the ultimate of each row the corresponding diagonal entry in the triangle yields the estimate of the reserves required for each year to meet the expected claim. Adding up the expected reserves for each gives the total estimate of reserves required to meet the claims.

There is a variant of the chain ladder technique in which multiplicative factors are constructed row-wise for the accident years instead of the column-wise development years. It turns out that reserves estimated by his methods are the same as those obtained using the development factors. A simple mathematical proof may be found in Cowell (2009).

3. THE NEW BOOTSTRAP METHOD

In the standard chain ladder technique, development factors are formed by taking the ratios of column sums in the cumulative claims triangle. For the local chain ladder bootstrap method we form just the ratios of neighbouring values in the cumulative claims distribution table. That is, we calculate the ratios

$$\lambda_{i,j+1} = D_{i,j+1}/D_{i,j},$$

for each pair of neighbouring values in the cumulative claims distribution table. This stage is only performed once.

The second step is to construct a bootstrap sample, in analogy to the chain ladder technique:

$$\hat{D}_{i,j+1} = \Lambda_{i,j+1} \hat{D}_{i,j}, \text{ for } j > n - i - 1,$$

where $\Lambda_{i,j+1} \in \{\lambda_{1,j+1}, \lambda_{2,j+1}, \cdots, \lambda_{j,j+1}\}$ is a randomly chosen value from the set of local ratios calculated for column $j+1$ of the claims triangle, and where we define the diagonal entries as for the standard chain ladder technique.:

$$\hat{D}_{i,n-i+1} = D_{i,n-i+1} \text{ for } i = 2, \ldots, n.$$

Having filled the cumulative claims distribution table using the randomly sampled local development factors, the ultimate claims for each row can be found in the same manner as for the chain ladder technique.

By repeatedly carrying out the second step on the lower diagonal, we obtain as big a bootstrap sample as desired, from which the distribution of the reserves for individual years and the distribution of total reserves may be empirically estimated, together with summary statistics such as mean and variance of reserves.

In analogy with the chain ladder technique there is a variant in which the local factors are found from row-wise ratios instead of column-wise ratios, and the table is completed stochastically using these row-wise factors. However, unlike the chain ladder technique, this procedure leads to a different distribution of reserves. A third variant which uses either a randomly chosen row or column factor when filling in an entry is also possible. These three variants will be called respectively *across*, *down* and *both* methods. Further details may be found in (Cowell 2009).

4. A SIMPLE SIMULATION STUDY

I carried out a simple simulation study in which a claims triangle was simulated from a model described by Schiegl (2004). This meant that the true distribution of reserves was known. The predictions of the bootstrap method were compared to this true distribution, and also against those of Mack's method (Mack 1993) and an over-dispersed Poisson model of England and Verrall (1999). The simulation steps were as follows:

1. Use the model to simulate an upper claims triangle.
2. Use the model to simulate 1000 completions of the lower triangle (to give a sample from the true distribution of the reserves).
3. Use the local chain ladder bootstrap method (all three variants), the over-dispersed Poisson method, and Mack's method, to estimate the distributions of reserves, using as input the upper triangle of Step 1 (by creating 1000 samples from each model).
4. Compare the true and estimated reserve distributions using quantile-quantile plots and box-plots.

The results of the simulation are summarized in the plots of Figure 1 showing the distribution of total outstanding reserves. The QQ-plot in the top left is made from splitting the data generated on Step 2 above to show that the simulated values are behaving properly, with the points lying close to the line of slope 1. The middle top QQ-plot shows the distribution estimated using Mack's method, and to the right of this the estimated using the over-dispersed Poisson model. Below these are the three bootstrap variants, from left to right the across, down and both methods. Also shown are comparative box-plots of the true distribution (left) and the various modelling estimates. What is apparent is that the variability of the non parametric bootstrap method is much smaller than Mack's method and the over-dispersed Poisson model, and much more in line with the true variability. All methods underestimate the true median or median by a similar amount. Other simulations (not presented here due to lack of space) have shown similar behaviour, with sometimes the median under-estimated and sometimes over-estimated (in the same direction by all methods), with usually but not always the "down" bootstrap variant producing variability estimates closer to the true variability.

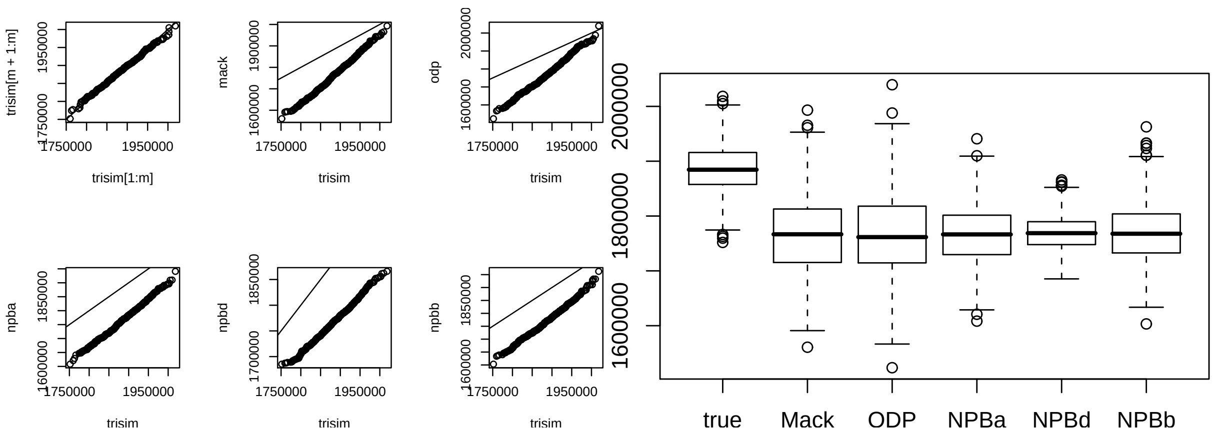

Figure 1: Results from a simulated triangle. The local chain ladder bootstrap appears to have variability closer to the true distribution, with the "across" and "both" variants performing better. All methods underestimate the mean.

5. SUMMARY

I have presented a simple and novel non-parametric bootstrap method, in three variant forms, for estimating the distribution of reserves given data in the form of a claims triangle

A simple simulation study showed that the local chain ladder bootstrap method tend to be biased in location of the distribution (median), but in an unpredictable way. Curiously the local chain ladder bootstrap method appears to be biased in the same direction and about the same amount as the over-dispersed Poisson method and Mack's method in each simulation

This suggests that if it could be understood when and by how much the bias is happening, and could be corrected for by a simple estimable translation, the local chain ladder bootstrap method could provide a practising actuary with reasonable simple and robust method for estimating the distribution of claims reserves.

More details of results from other simulations, and also comparative estimates for some historical data, may be found in Cowell (2009).

Finally, a simple program (running under Microsoft Windows) for carrying out the analysis of the bootstrap method using either supplied data, or for simulating triangles using Scheigel's model, may be downloaded from the software page of the author's web page at `http://www.staff.city.ac.uk/~rgc`, which is free for research and non-commercial use only.

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A FRAMEWORK FOR PRICING A MORTALITY DERIVATIVE: THE q-FORWARD CONTRACT

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1. INTRODUCTION

Traditionally the capital markets limited their role in insurance to provide capital in the form of shares and bonds to insurance and reinsurance companies. However, in recent years a whole new range of insurance linked products has become available. In this work we focus on an insurance-linked derivative which is a derivative whose value is linked to an insurance index (or company specific losses) rather than a stock price. Longevity and mortality derivatives are financial contracts that allow market participants to either take exposure, or hedge exposure, to the longevity and mortality experience of a given population of individuals.

In the past century remarkable improvements in human life expectancy have been observed. However, future demographic patterns are uncertain and difficult to be predicted accurately. The uncertainty affecting such trends is referred to as longevity risk. Longevity risk derives from systematic deviations of the number of death from its expected value; it is a macro risk, or systematic risk, which cannot be reduced by diversification. A good understanding of mortality rate patterns over time is needed, so that the underlying changes can be accurately modelled and projected into the future. A failure to allow appropriately for longevity risk would mean that the premiums and reserves for annuity and pension products would be understated with potentially disastrous consequences for governments and financial institutions involved. Recently, on the one hand there is an increasing emphasis on the market value of longevity risk in a regulatory context (Solvency II, IFRS), where authorities are focusing on capital adequacy to face the adverse impact of the risk and pension and annuities providers need to hedge their exposure to longevity. To have an idea of the impact of longevity risk, in 2007 the UK pension regulator estimated that the present value of UK's pension fund liabilities increases by 3% per additional year of life expectancy. On the other hand, there is an increasing appetite from investors for mortality linked securities. To satisfy both hedgers and investors, several investment banks are building the technology for trading longevity risk. In particular, according to the OECD report (2010), the potential size of the longevity market is valued of the order of \$ 25 trillion. In other words, longevity becomes a new asset class for different stakeholders like Insurance Companies, Pension Buyout Funds, Hedge Funds and so on. For such institutional investors, longevity represents a potentially attractive investment opportunity primarily because it is not correlated to non-life, credit and market risks. Investors see longevity as a new asset class providing good diversification due to low correlation with other assets and positive risk premium. In this perspective, capital markets can transfer longevity exposure and start to promote the development of a liquid traded market in longevity risk transfer. Blake and Burrows (2001) were the first to advocate the use of mortality-linked securities to transfer longevity risk to the capital markets. Their proposal has generated considerable attention in the last few years, and major investment banks and reinsurers are now actively innovating in this space (see Blake et al. (2008), for an overview). To break down the barriers to market growth, capital markets support the development of consistent standards, methodologies and benchmarks, for building a liquid trading market, particularly in UK through the Life & Longevity Markets Association (LLMA). The standardisation process involves a default methodology, in terms of full and detailed disclosure of data sources, algorithms, rules, degree of discretion and governance procedures, and provides a template that market participants can use to develop credible, robust customised longevity indices that facilitate longevity transactions. Derivatives on mortality index can be developed to transfer longevity risk. In order to price this kind of products, many factors have to be kept into account: the structure of the product, including pension amount, payment frequency, details on guaranteed payments; data on the reference lives and base mortality rates, the starting point for analyzing the possible evolution of future cash flows; the expected mortality improvement, important for estimating the possible evolution of the future cash flows and the risk premium.

In this extended abstract, we focus on derivatives involving the exchange of the realized mortality rate of a population at some future date, in return for a fixed mortality rate agreed at inception: the q-forwards. We present the stochastic Lee Carter model for projecting mortality and finding the best estimates of the mortality probabilities necessary to define the fixed leg of the q-forward. In D'Amato et al. (2011) we deepen the analysis and discuss the possible uses of financial tools for pricing and managing, mortality-dependent contracts. In particular, we make use of the similarities between mortality and financial setting to show how we can model mortality risks and price mortality-related instruments using adaptations of the pricing frameworks that have been developed for financial derivatives.

2. THE MODEL

The q-forward contract is the simplest instrument for transferring longevity risk. A q-forward is an agreement between two counterparties to exchange at a future date an amount equal to the realized mortality rate of a given population (the floating leg) at that future date, in return for a fixed mortality rate (the fixed leg) agreed upon at the inception of the contract. The floating leg of the instrument references the uncertain future mortality rate of the population, as reflected by an appropriate index. A counterparty hedging longevity risk will receive fixed and pay floating mortality. A q-forward can be stipulated between a pension fund or annuity provider (protection buyer) and a life insurer provider (protection seller). Initially the value of the contract is zero as the present values (PV) of floating and fixed legs are equal. If the realized mortality is less than expected the protection buyer receives a net payment.

LLMA provides a simple framework to price this kind of contract: starting from the base mortality table, they consider mortality improvements in terms of a given percentage of the historical mortality rates. D'Amato et al. (2011) show the impact of longevity risk on the pricing of q-derivatives projecting the mortality improvements in a more accurate way. To this aim, we consider a stochastic model for mortality projections. A large number of projection models are available to generate future mortality rates from historical death. Such models include the Lee-Carter (LC) model (1992), widely considered because it produces fairly realistic life expectancy forecasts, which are used as reference values for other methods. In recent years, there have been several extensions of the standard LC method; Renshaw and Haberman (2003) have shown an improvement in the fitting using a Poisson iterative version of the Lee-Carter. Our main contribution is to exploit this mortality projection model in the pricing of q-forward, improving the algorithm used to define the fixed leg of the contract.

The model can be summarized as in the following:

$$\mathrm{E}[D_{x,t}] = d_{x,t} = \text{Poisson}(E_{x,t}\mu_{x,t})$$

where $D_{x,t}$ is the number of deaths at age x and time t, $E_{x,t}$ is the exposure at risk and $\mu_{x,t}$ is the death rate, where

$$\mu_{x,t} = \exp(\alpha_x + \beta_x \kappa_t).$$

The parameters of the model are estimated by maximising the Poisson likelihood function. In order to fit the Poisson log-bilinear model, we resort to the iterative fitting method as described in Renshaw and Haberman (2003). According to this method, it is possible to optimise the Poisson likelihood by monitoring the associated deviance:

$$D(d_{x,t}, \hat{d}_{x,t}) = \sum_{x,t} \text{dev}(d_{x,t}, \hat{d}_{x,t}) = \sum_{x,t} 2\{d_{x,t} \log(\frac{d_{x,t}}{\hat{d}_{x,t}}) - (d_{x,t} - \hat{d}_{x,t})\},$$

where $\hat{d}_{x,t} = E_{x,t} \exp(\hat{\alpha}_x + \hat{\beta}_x \hat{\kappa}_t)$.

3. NUMERICAL APPPLICATION

We have produced an application to the Italian population. We have considered a q-forward with notional amount equal to 10 000 euro, with trade date 31 December 2006 and maturity date 31 December 2017. The population data considered are those collected in the Italian mortality table from 1974 to 2007 downloaded from the Human Mortality Database. The reference population is represented by Italian males aged between 70 and 74 in 2007; the aggregate mortality rate for the portfolio is an average of the mortality rates for each of the individual ages, see Table 1.

Age	$q(2007)$
70	2.02195%
71	2.29090%
72	2.56166%
73	2.84514%
74	3.19871%
mean	2.58367%

Table 1: Base mortality rate for the reference population

Let $q_{FE}(2007:2017)$ be the fixed forward mortality rate and let $q(2017)$ the floating rate, given by the mortality rate in 2017. In order to determine the fixed payment of the q-contract, we have to consider that the longevity market is net short; thus investors require compensation to take on longevity risk. For this reason, forward mortality rates should be lower than expected mortality to provide a risk premium. For illustrative purpose, LLMA quotes the fixed mortality rates in terms of the best estimate mortality rate according the following formula:

$$q_{FE}(2007:2017) = q_{BE}(2017)(1-r)^t$$

where q_{BE} is the best estimate mortality rate and r is a given risk premium.
In our application we have calculated the best estimate mortality rate projecting the mortality through the iterative Poisson Lee Carter model. Table 2 shows the results.

Age	$q_{BE}(2017)$
70	1.81760%
71	2.02330%
72	2.24682%
73	2.50887%
74	2.78085%
mean	2.27549%

Table 2: Best Estimate mortality rate

If there were no risk premium for transferring longevity risk, then the best estimate mortality rate would correspond to the fixed rate in the q-forward transaction. However, the longevity market is net short; thus investors require compensation to take on longevity risk. For this reason, forward mortality rates should be lower than expected mortality. Different approaches can be used to include the risk premium in the pricing. For illustrative purpose, LLMA quotes the risk premium in terms of a given decrease in the level of the best estimate mortality rate. In D'Amato et al. (2011) a different way of defining the risk premium is presented.

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PROFIT TEST MODEL FOR PENSION FUNDS USING MATRIX-ANALYTIC MODELING

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In the present work we apply matrix-analytic methods to build up a pension fund model with our main objective being to describe the profit arising from the launch of a pension plan. We chose the Present Value to describe the profitability of the new project because it is a widely used and transparent measure, as shown in Smart (1977). Present Value consists of discounted cash inflows and outflows which we obtain for every individual at future times. The problem requires that we calculate tariffs and give a proper description of the evolution of plan participants before and after the stabilization moment, which we also need to determine.

Here, we consider a defined benefit pension plan with a lump-sum payment upon retirement and the sum of accrued premiums in case of a death or upon leaving the plan for other reasons.

The evolution of plan participants is assumed to follow a Markov chain with four sets of states: active participant, retired participant, dead participant and participant stopping the contract. Retired and dead participants, and those stopping their contracts are replaced by new participants. The policy, whereby the replacements take place is a part of the model and composed of a delay and a structure of the replacement. We need to estimate the transition rates of the Markov chain and, in particular, the rates at which an individual dies, retires or stops the contract.

In order to determine Cash inflow and Cash outflow we need to determine these properly for every individual. To do this we look for the distribution of the number of years already spent in the plan by an active participant, and the succession of phases which he visited in the past.

This type of models may be used with more than one objective in mind. For instance, it allows us to find values of pension plan characteristics to increase the profit. In the present work we look for the optimal replacement speed to balance future cash flows in the long-run.

1. EVOLUTION OF PLAN PARTICIPANTS

We use Markov chains to describe the evolution of pension plan participants. This approach was used by Bertschi et al. (2003) and by Mettler (2005), where an evolution of plan participants is described by a generator matrix. We assume that phases for participants are grouped into four categories: active (A), retired (R), surrendered (S) and dead (D), and denote the conjunction of phases for R, S and D as Q. Denote the phase at time t as Φ_t. We look at the evolution in continuous time and organize the structure of the generator matrix according to these four sets of phases. Therefore, we assume the generator matrix to have the form

$$\Pi = \begin{pmatrix} \Pi_{AA} & \Pi_{AR} & \Pi_{AS} & \Pi_{AD} \\ \Pi_{RA} & \Pi_{RR} & 0 & 0 \\ \Pi_{SA} & 0 & \Pi_{SS} & 0 \\ \Pi_{DA} & 0 & 0 & \Pi_{DD} \end{pmatrix}$$

where Π_{AA} is an n by n matrix that describes active participants, Π_{AR}, Π_{AS} and Π_{AD} are column vectors of size n representing transitions to one of the quitting states, Π_{RA}, Π_{SA} and Π_{DA} are row vectors of size n describing replacements in case of retirement, surrender and death, correspondingly. We assume Π_{RR}, Π_{SS} and Π_{DD} to be scalars, which are equal to each other and represent the rates at which an individual who leaves the plan is being replaced.

We calculate all financial results once in a year, meaning a discrete evolution of plan participants:

$$P_{t+1} = P_t e^{\Pi}, \quad 0 \le t < H,$$

where P_t is the distribution of Φ_t at time t, $P_t^{(i)} = \mathbf{P}[\,\Phi_t = i\,]$, and H is a chosen time horizon. Denote the number of years spent in the system at time t as an active participant as Ψ_t. The probability that a new participant starts his contract at time t in phase i is

$$M_t^{(i)} = \mathbf{P}[\,\Psi_t = 0\,,\ \Phi_t = i\,]\,,$$

$i \in A$, and it is determined as $M_t^{(i)} = P_t^{(Q)} \left(e^{\Pi}\right)_{Qi}$. Let us denote $\mathbf{M}_t = \left(M_t^{(i)},\ i \in A\right)$.

Transition rates: aging and death

To find the death rates, we use an approach introduced by Lin and Liu (2007). According to this approach, when active, participants follow a continuous time aging process, where time of death follows a phase-type distribution (see Latouche and Ramaswami (1999)) with generator matrix L and initial probability vector $\alpha = [1\ 0\ \ldots\ 0]$. Therefore, the aging process is a finite-state Markov process where the states are defined as health indices called “physiological ages”. The probability to stay alive at least t years for a newborn individual (survival probability) is equal to

$$S^A(t) = \alpha e^{Lt}\mathbf{1}.$$

The phase-type distribution may be fitted to actual mortality data and may also be adapted to different assumptions.

Transition rates: retirement

In order to determine the retirement rates we assume that retirement happens at the statutory retirement age R. In order to find the retirement rates for every “physiological age”, we define a model similar to the aging model described above, but with an additional absorbing state representing retirement. The probability to stay alive at least t years for a newborn individual (survival probability) is denoted as $S^{AR}(t)$ and has a representation similar to the one for the aging model. The generator matrix L^R of this modified model is obtained from the generator L by subtracting retirement rates r from the diagonal. Define the jump function $F = F(t)$ such that

$$\begin{cases} F(t) = S^A(t), & t \in [0, R) \\ F(t) = 0, & t \geq R \end{cases}$$

In order to find r we need to find an approximation of the function $F(t)$ by a phase-type distribution. Denote the expected "physiological age" at age R as i^*. We assume the vector r to be a constant value r_1 for all phases less than i^* and a constant value $r_1 + r_2$ otherwise, and we apply the least squares method as a fitting procedure.

Transition rates: surrender option

Surrender rates w ($w_j,\ j \in A$) for active employees should be chosen with caution and match the actual data. From the data we can obtain the empirical distribution of the number of years spent in the pension plan at the moment of stopping the contract. In our experience, the form of this distribution is quite stable for all pension plans. Therefore, we can derive an empirical survival probability $\hat{S}_t$, which is the fraction of the plan participants who stay more than t years as active participant.

In order to find a surrender rate for every physiological age we define a model similar to the retirement model described above, but with an additional absorbing state representing surrenders. The generator matrix becomes $L^S(\mathbf{w}) = L^R - diag(\mathbf{w})$. The main difference of this model with respect to the previous two is that time t now has the meaning of the number of years spent in the pension plan and no longer the age of a participant. To adjust the interpretation, we take into account the initial distribution of physiological ages of the plan participants, P_0. With this adjustment the term $\left(P_0 e^{L^S t}\right)_j$ is the probability to survive at least t units of time in the pension plan as an active participant and to be in phase j at time t. In order to determine $\mathbf{w}$, we need to find an approximation of the empirical probability $\hat{S}_t$. As in the problem of finding retirement rates we assume a fixed structure of the vector $\mathbf{w}$ and apply the least squares method to the fitting of $P_0 e^{L^S(\mathbf{w})t}\mathbf{1}$ to $\hat{S}_t$ for all t.

A rather good approximation can be obtained assuming the vector $\mathbf{w}$ to hold a similar structure as for retirement rates. However, in this case i^* has a different meaning. Denote the maximum possible length of service in the pension plan obtained from the data as l^*. Then i^* is the expected phase of an individual who spent l^* years in the system with retirements described above.

In order to compare all three models we need to construct a survival function for the model with surrenders. We cannot obtain it in the same way as for the first two models due to the difference in the interpretation of the time in the model. In the first two cases, at the start, all the people are newborn and the time is connected to the age of an individual, whereas in the third case the time starts at the moment of entering the pension program. In order to perform a proper comparison we construct the survival function such that it takes into account the ages of the plan participants prior to the moment of entering the plan:

$$S^{ARS}(t) = 1 - \left(\sum_{x=0}^{t} \left(1 - \alpha e^{Lx} e^{L^S(t-x)}\right) \mathbf{P}\,[\, B = x\,] + \left(1 - \alpha e^{Lx}\mathbf{1}\right) \sum_{x=t}^{100} \mathbf{P}\,[\, B = x\,] \right),$$

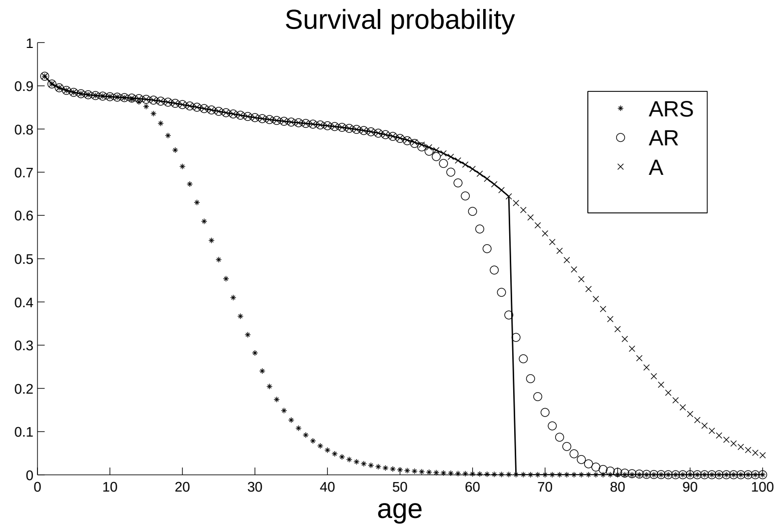

Figure 1: Survival probability.

where $P[\ B = x\ ]$ is the initial distribution of real ages of the plan participants. All the survival functions S_t^{ARS}, S_t^{A} and S_t^{AR} are presented in Fig. 1 in different types of dots. The solid line represents the jump at the retirement age $R = 65$.

Transition rates: replacements

We need to make assumptions on a structure and a speed of the replacement. We assume the structure to be identical to the initial distribution of participants in the system and the same replacement speed for surrendered, retired and dead participants.

2. CALCULATION OF TARIFFS

We use a traditional balance approach to calculate the tariffs for the chosen pension plan. We assume that an individual in phase j at the moment t has the salary $\Theta_t^{(j)}$, and calculate the tariff as a percentage of this salary. We also assume the fund to have two types of expenditures per policy: annual, c, and initial, I. In the balance equation we take into account two decrements: death and surrender option. However, the model we chose to describe the evolution of the plan participants poses some difficulties. Both of them are caused by the presence of "physiological ages" instead of real ages. First of all, it makes the distance to the statutory retirement age undefined, which implies an undefined horizon for calculations. Secondly, for an individual in phase j, the probability to survive within t years is no longer a multiplication of successive one-year survival probabilities for the phases from j to $(j + t - 1)$. To deal with these difficulties, we calculate tariffs $\mu = \mu(x, j)$ for every phase j and every age x:

$$\mu(x,j)\ddot{a}^{[\theta]}_{j:R-x} = {}_{R-x|}\ddot{a}^{[\theta]}_{j} + \mu(x,j)A^{[\theta]}_{j:R-x} + c\mu(x,j)\ddot{a}^{[\theta]}_{j:R-x} + I,$$

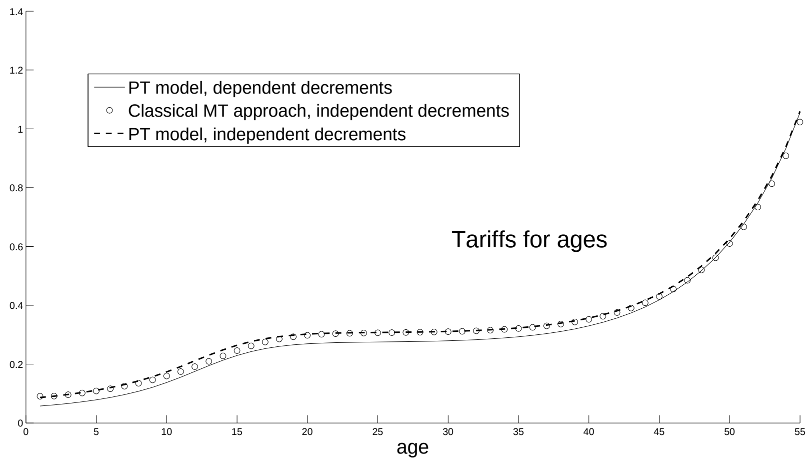

Figure 2: Tariffs for ages.

where

$$_{n|}\ddot{a}_j^{[\theta]} = \sum_{k=0}^{\infty} v^{n+k}\ {}_{n,k}p_j^{[SD,D]}\ , \quad \ddot{a}_{j:n}^{[\theta]} = \sum_{k=0}^{\infty} v^k\ {}_{k,0}p_j^{[SD,D]}\ \Theta_k^{(j)},$$

$$A_{j:n}^{[\theta]} = \sum_{k=1}^{n} v^k\ {}_{k-1}\,(p_q)_j^{[SD]} \sum_{i=0}^{k-1} \Theta_i^{(j)}.$$

In the equation above, ${}_{n,k}p_j^{[SD,D]} = \alpha^{(j)} e^{L^{SD}n} e^{Lk}\mathbf{1}$ is the probability to remain active in the plan for n years with respect to death and surrender option and then remain alive for k years. The quantity ${}_{n}\,(p_q)_j^{[SD]} = \alpha^{(j)} e^{L^{SD}(n-1)}(1 - e^{L^{SD}}\mathbf{1})$ represents the probability to remain active for n years and then become inactive due to one of the indicated reasons. Here, $\alpha_i^{(j)} = 0$, if $i \neq j$ and $\alpha_i^{(j)} = 1$, if $i = j$; $L^{SD} = L - diag(\mathbf{w})$.

We also consider another method of tariffs valuation. In the circumstances where one has two separate sets of survival probabilities, one for death, one for surrender, it is a standard practice to make the approximation that the two decrements are independent. In order to calculate tariffs with this method it is sufficient to use the survival probabilities in the standard balance equation with two independent causes of decrements. The method is not as precise as the previous one, however, it gives similar results, we give an illustrative example in Fig. 2.

To obtain the tariff for the real age x we weight $\mu(x, j)$ with probabilities to be in the phase j conditioned on the age being x. Verification of the tariffs is based on the comparison with the tariffs calculated with the corresponding mortality table and where the decrements are assumed to be independent. The results are presented in Fig. 2.

In order to derive the tariff $\mu(j)$ for phase j we perform a weighted sum of tariffs $\mu(x, j)$ over all x, multiplied by the probability to be in the age x, conditioned on the phase being j.

3. TIME TO STABILITY

Time to stability t^* is the length of time after which all characteristics of the population become stable. This time is useful to know when choosing the time horizon for the cash flows calculations. In the stationary regime the matrix $e^{\Pi t}$ has all zero eigenvalues and one eigenvalue which is equal to one. The matrix e^{Π} has one eigenvalue which is equal to one and others that are strictly less than one. Taking these facts into account we find t^* from the equation $\lambda^{t^*} = \varepsilon$, where λ is the second maximal eigenvalue of the matrix e^{Π} in terms of modules and ε is a required degree of precision, so that $e^{\Pi t}$ is nearly constant for $t > t^*$.

4. CASH FLOWS AND PRESENT VALUES

To estimate the profitability of the fund with respect to the chosen pension plan, we need to obtain its future cash flows. Clearly, at every moment of time, the total cash flow is equal to the sum of the cash flows over all individuals. To properly calculate the cash flow coming from an individual in phase j at time t, we need to know how long the individual has already been in the system and which phase he was in at the moment of entering the program. Thus, for every individual we need to find the distribution of years spent in the system (called the "seniority distribution") and the distribution of the entering phase (called the "reversal probability"). Denote the seniority distribution vector at time t as ${}^r\mathbf{N}_t = \left({}^rN_t^{(i)},\ i \in A\right)$, where ${}^rN_t^{(i)} = \mathbf{P}\left[\ \Psi_t = r\ ,\ \Phi_t = i\ \right]$ is the probability that a participant at time t has physiological age i and seniority r in the plan. As suggested by Janssen and Manca (1994), we can solve the system of equations:

$$\begin{cases} {}^r\mathbf{N}_{t+1} = ({}^{r-1}\mathbf{N}_t)\, e^{\Pi_{AA}} \\ {}^0\mathbf{N}_{t-r+1} = \mathbf{M}_{t-r+1} \end{cases} \quad , \quad r \le t+1$$

For the cash flow calculation we need conditional probabilities ${}^S_r\mathbf{P}_i(t) = \mathbf{P}\left[\ \Psi_t = r \mid \Phi_t = i\ \right]$ for active plan participants, which we find from the equation

$${}^S_r\mathbf{P}_i(t) = {}^rN_t^{(i)} \ /\mathbf{P}\left[\ \Phi_t = i\ \right].$$

The reversal probability to enter the pension plan r years ago in phase j given the phase i at time t we define as ${}^R_r\mathbf{P}_{ji}(t)$. This probability can be found from the equation

$$\begin{aligned} {}^R_r\mathbf{P}_{ji}(t) &= \mathbf{P}\left[\ \Psi_{t-r} = 0\ ,\ \Phi_{t-r} = j\ ,\ \Psi_t = r\ ,\ \Phi_t = i\ \right] / \ {}^rN_t^{(i)} \\ &= \mathbf{P}\left[\ \Psi_{t-r} = 0\ ,\ \Phi_{t-r} = j\ \right] \left(e^{\Pi_{AA} r}\right)_{ji} / \ {}^rN_t^{(i)}\ , \end{aligned}$$

where $\left(e^{\Pi_{AA} r}\right)_{ji}$ is the probability to stay among actives for r years starting from phase j and being in phase i at the end of the period. In terms of the distribution of new plan participants, ${}^R_r\mathbf{P}_{ji}(t) = M_{t-r}^{(j)} \left(e^{\Pi_{AA} r}\right)_{ji} / \ {}^rN_t^{(i)}$.

We assume all the premiums to be paid at the beginning of each year and we denote the cash inflow coming at time t from the individuals in phase i, $i \in A$, as $F_t^+(i)$, where

$$F_t^+(i) = N^o \mathbf{P}[\ \Phi_t = i\ ] \left[{}_0^S\mathbf{P}_i(t)\ \Theta_t^{(i)} \mu(i) + \sum_{r=1}^{t} {}_r^S\mathbf{P}_i(t) \sum_{j \in A} {}_r^R\mathbf{P}_{ji}(t)\ \Theta_t^{(j)} \mu(j) \right].$$

N^o is the total number of plan participants. The total cash inflow in the year t is $F_t^+ = \sum_{i \in A} F_t^+(i)$. The cash outflow coming at time t from the individual in phase i, $i \in A$, consists of several terms:

$$F_t^-(i) = N^o \mathbf{P}[\ \Phi_t = i\ ] \left(e^{\Pi}\right)_{iR} + cN^o \mathbf{P}[\ \Phi_t = i\ ] + N^o (c + I) M_t^{(i)} +$$

$$+ N^o \mathbf{P}[\ \Phi_t = i\ ] (\left(e^{\Pi}\right)_{iS} + \left(e^{\Pi}\right)_{iD}) \sum_{r=0}^{t} {}_r^S\mathbf{P}_i(t) \sum_{j \in A} r\ {}_r^R\mathbf{P}_{ji}(t)\ \Theta_t^{(j)} \mu(j).$$

Here the first term represents payments to newly retired participants; the second and the third term are periodic expenditures for current active policies and periodic and initial expenditures for new policies; the last term describes payments to surrendered and dead participants.
We derive the present value implementing the formula $PV_T = \sum_{t=0}^{T} \left(F_t^+ - F_t^-\right) / (1 + v)^t$, where v is a discount rate.

5. OPTIMAL MODIFICATIONS

A first question about the model is how the speed of finding new clients affects the cash flows. To answer this question we solve the equation

$$\widehat{F}^+(\lambda) = \widehat{F}^-(\lambda),$$

where $\widehat{F}^+$, $\widehat{F}^-$ are the values of the cash inflows and outflows in the long run and $1/\lambda$ is the speed of the replacement for surrendered, retired and dead plan participants. In order to calculate $\widehat{F}^+$, $\widehat{F}^-$ we obtain stationary characteristics of the population of the plan participants. The logic of its derivation as well as the derivation of $\widehat{F}^+$, $\widehat{F}^-$ remains the same as in the previous sections, assuming the initial distribution to be the stationary distribution of the population.

We solve this balance equation numerically. The resulting difference in cash flows is presented in Fig. 3. The X-axis is the number of years representing the delay for the replacement. The Y-axis is the difference between cash inflow and cash outflow. The starred part of the curve corresponds to a positive future cash flow and the solid line – to its negative values. In this example all salaries are fixed and equal to one. The behavior of the curve is quite logical – the faster the replacement, the greater the money. In order to have a positive profit, the replacement should happen within about 3.3 years.

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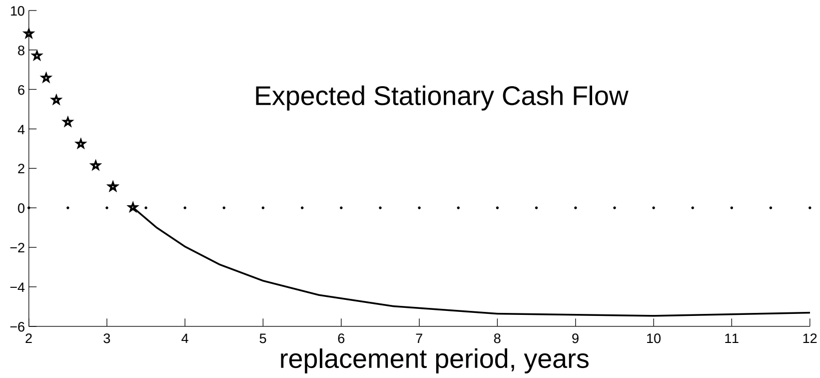

Figure 3: Expected Stationary Cash Flow.

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ON OPTIMAL REINSURANCE CONTRACTS[1]

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Optimality results in the reinsurance literature focus mainly on the cedent's perspective. When the reinsurer's perspective is not part of the considerations often Stop-Loss contracts are identified as optimal. Current papers (see e.g. Bernard and Tian (2009) and Cheung (2010)) study optimal reinsurance contracts that minimize the VaR or TVaR of the cedent. It turns out that again the "classical" reinsurance treaties, as Stop-Loss and Quota-Share lead to optimal results. Some recent results on this subject lead to optimal reinsurance contracts, that do not have linear or constant retention functions. See e.g. Kaluszka (2005), where it is shown, that the optimal retention function is of a logarithmic type, if Wang's principle is applied for the calculation of the reinsurer's premium.
In practice, reinsurance companies will often avoid such Stop-Loss contracts or set upper limits, partly to reduce the problem of careless claim settlements and potential moral hazard of the first-line insurer.
The goal of this study is to take this problem into account more explicitly and to optimize the situation for both parties, the cedent and the reinsurer, where the objective function is a linear combination of expected utility of the cedent and the reinsurer, respectively. Some analytical and numerical results are provided.

1. INTRODUCTION

We consider a general class of reinsurance treaties, where the cedent and the reinsurer share the risk X according to the following rule:

$$C = \begin{cases} X, & X \le x^* \\ \min(X, h(X)), & X > x^* \end{cases} \quad \text{and} \quad R = \begin{cases} 0, & X \le x^* \\ \max(X - h(X), 0), & X > x^* \end{cases},$$

where x^* is determined such that $x^* = h(x^*)$. This restriction is introduced to assure continuity of the reinsurance treaty.

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We analyze the linear combination of the expected utilities of cedent and reinsurer, respectively and study the form of the function $h(x)$ that maximizes the following problem:

$$\max_{h(x)} \left\{ \alpha \mathbb{E}(u_R(w_R + P_R - R)) + (1-\alpha)\mathbb{E}(u_C(w_C + P - P_R - C)) \right\},$$

where u_R and u_C are utility functions of the cedent and reinsurer, respectively and with w_R and w_C we denote the initial capital. Furthermore we denote by P and P_R the original and the reinsurance premium of the risk X. The risk X is distributed with some distribution function $F(x)$.

We first study the case where the reinsurance premium is calculated under the expected value principle, i.e. the security loading is proportional to the expected value:

$$P_R = (1 + \Theta_R) \int_{x^*}^{\infty} (x - h(x)) f(x) dx$$

Secondly we analyze the situation under the variance principle, i.e. the security loading is proportional to the variance.

$$P_R = \int_{x^*}^{\infty} (x - h(x)) f(x) dx + \Theta_R \left(\int_{x^*}^{\infty} (x - h(x))^2 f(x) dx - \left(\int_{x^*}^{\infty} (x - h(x)) f(x) dx \right)^2 \right).$$

We restrict ourselves to the case where P_R is fixed. In this setting the cedent specifies the premium level that he can afford to pay for the reinsurance. We optimize therefore over all contract forms that lead to this premium P_R. The parameters of $h(x)$ can be determined through this assumption.

2. THE PERTURBATION APPROACH

Our aim is to find a reinsurance contract, more precisely a function $h(x)$, that maximizes

$$\max_{h(x)} \Bigg\{ \alpha \left(\int_0^{x^*} u_R(w_R + P_R) f(x) dx + \int_{x^*}^{\infty} u_R(w_R + P_R - x + h(x)) f(x) dx \right) + \\ + (1-\alpha) \left(\int_0^{x^*} u_C(w_C + P_C - x) f(x) dx + \int_{x^*}^{\infty} u_C(w_C + P_C - h(x)) f(x) dx \right) \Bigg\}.$$

We follow the considerations in Chan and Gerber (1985) and apply the following perturbation approach:
We assume $h(x)$ to be optimal and set the perturbed version $\tilde{h}(x)$ to $\tilde{h}(x) = h(x) + t\, g(x)$, where $g(x)$ is some arbitrary function. The function $m(t)$, with

$$m(t) = \alpha \mathbb{E}(u_R(w_R + P_R - (X - \tilde{h}(X))_+)) + (1-\alpha)\mathbb{E}(u_C(w_C + P_C - \min(X, \tilde{h}(X))_{X > x^*})),$$

obtains its maximum then at $t = 0$. The expressions $(X - \tilde{h}(X))_+$ and $\min(X, \tilde{h}(X))_{X > x^*}$ denote the cedent's and the reinsurer's part of the risk under the perturbed reinsurance treaty $\tilde{h}(x)$.
We then calculate the derivative of $m(t)$ w.r.t. t and set the derivative $m'(t)$ to 0, as $t = 0$. Due to the structure of $m(t)$, the derivative $m'(t)$ is again a function in terms of expected values. Inside

the expectation we can rewrite all terms, such that we get $\mathbb{E}(g(x)(\ldots)) = 0$.
Since the integration domain of the expected value is the positive real half-axis and $g(x)$ is an arbitrary function, the expression inside the brackets $(\ldots)$ has to be 0.

We apply the above described approach now in the case, where the reinsurer's premium is calculated according to the expected value principle. The derivatives w.r.t. t of P_R and $P_C = P - P_R$ can then be written as

$$\frac{\partial}{\partial t}P_R = (1+\Theta_R)\int_{x^*}^{\infty}(-g(x))f(x)dx, \qquad \frac{\partial}{\partial t}P_C = (1+\Theta_R)\int_{x^*}^{\infty} g(x)f(x)dx.$$

This leads to an integral equation in $h(x)$. Solving the integral equation, we obtain an implicit equation for $h(x)$:

$$\begin{aligned} H(x,h(x)) =& u'_C(w_C + P_C - h(x)) - (1+\Theta_R)d^*\hat{i}^* - \frac{\alpha}{1-\alpha}u'_R(w_R + P_R - x + h(x)) \\ &+ (1+\Theta_R)d^*\frac{\alpha}{1-\alpha}u'_R(w_R + P_R)F(x^*), \end{aligned}$$

where d^* and $\hat{i}^*$ are constants depending on the initial capital and the security loading of the reinsurer. Using the Theorem on implicit functions, the derivative $h'(x)$ can be obtained by

$$\frac{\partial}{\partial x}h = -\frac{\frac{\partial}{\partial x}H}{\frac{\partial}{\partial h}H} = \frac{\frac{\alpha}{1-\alpha}u''_R(w_R + P_R - x + h(x))}{u''_C(w_C + P_C - h(x)) + \frac{\alpha}{1-\alpha}u''_R(w_R + P_R - x + h(x))}.$$

With the constraint $x^* = h(x^*)$ we can determine x^* by the following implicit equation that can be solved numerically

$$x^* = w_C + P_C - u_C^*\left(\frac{\alpha}{1-\alpha}u'_R(w_R + P_R)(1 - (1+\Theta_R)d^*F(x^*)) + (1+\Theta_R)d^*\hat{i}^*\right),$$

where u_C^* is the inverse function of u'_C.

3. NUMERICAL RESULTS

We study the form of the function $h(x)$ now in two different settings: First we analyze the case of a risk-neutral utility function u_R. Secondly we derive the differential equation of $h(x)$ in the case of a risk-averse utility function, when the variance principle is applied for the calculation of the reinsurance premium P_R.

3.1. Risk-neutral u_R and expected value principle

Given the utility functions $u_C(x) = -e^{-\beta_C x}$ and $u_R(x) = x$, the non-linear differential equation for $h(x)$ reduces to

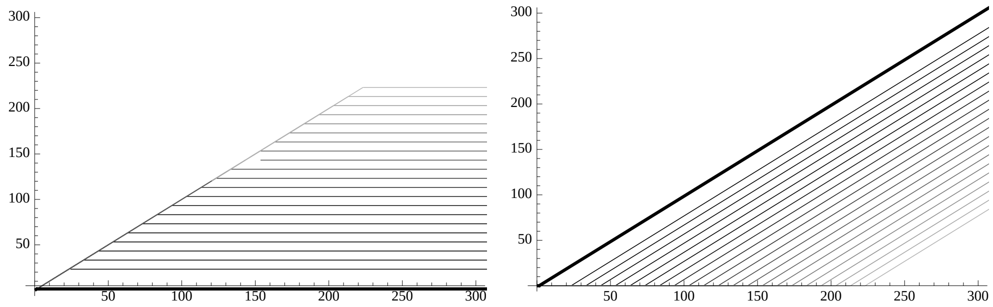
$$e^{\beta_C h(x)}h'(x) = 0.$$

The optimal reinsurance treaty is therefore a Stop-Loss contract, with

$$h(x) = \frac{1}{\beta_C} \ln \left(\frac{(1+\Theta_R)i^* - c^*}{F(x^*) - \Theta_R(1-F(x^*))} \right),$$

where the constants i^* and c^* depend on the initial capital and the weight α in the linear combination. With the restriction $x^* = h(x^*)$, one can determine the optimal x^* explicitly.

We analyze the optimal deductible x^* now for various levels of initial capital w_C and compare it with the classical Stop-Loss contract with retention x^{SL} (thick black line) that leads to the same premium.
The other parameters are set to $w_R = 120$ and $\beta_C = 0.3$. The initial capital w_C takes values in the set $\{20, 30, \ldots, 220\}$. The color gradient runs from dark gray for $w_C = 20$ to light gray for $w_C = 220$.



(a) Cedent's part: x^* vs. x^{SL} (b) Reinsurer's part: $x - x^*$ vs. $x - x^{SL}$

Figure 1: Cedent's and reinsurer's part part for different levels of w_C

Guerra and Centeno (2008) show that under the expected value principle a Stop-Loss contract maximizes the cedent's expected utility. The type of the optimal reinsurance contract does not change when introducing a risk-neutral reinsurer to the optimization problem. But we clearly see that the optimal retention increases for increasing w_C, meaning that the cedent has to cover a higher amount of claims than in the "cedent-only" case.

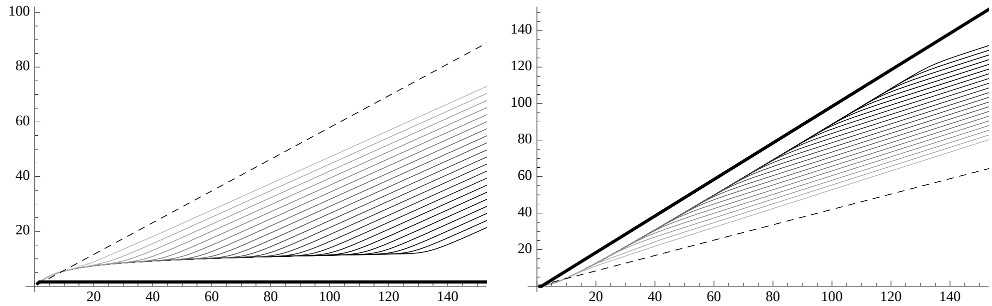
3.2. Risk-averse u_R and the variance principle

We assume exponential utility functions for both parties, i.e. $u_C(x) = -e^{-\beta_C x}$ and $u_R(x) = -e^{-\beta_R x}$ and furthermore we assume the variance principle for the calculation of P_R. Applying the perturbation approach as described above leads to an integral equation, that can be solved in an iterative way. In every iteration step we then solve the following non-linear differential equation

$$h'(x) + h(x) \frac{\beta_C e^{\beta_C h(x)}}{\hat{c}\beta_R e^{\beta_R(x-h(x))} - 2\Theta_R(\hat{d} - \lambda)} = 1,$$

where λ is determined in each iteration step and can be used to measure the accuracy of the obtained solution. The constants $\hat{c}$ and $\hat{d}$ depend again on the initial capital and the weight α in the linear combination. For a deeper discussion on the solution of the integral equation, see e.g. Polyanin and Manzhirov (1998).

We analyze the optimal reinsurance treaty $h(x)$ again for various levels of initial capital w_C and compare it with a Stop-Loss with retention x^{SL} (thick black line) and a Quota-Share contract with quota x^{qs} (dashed black line) that lead to the same premium P_R.
We fix the other parameters to $w_R = 120$, $\beta_C = 0.5$ and $\beta_R = 0.47$. The initial capital w_C takes values in $\{10, 15, \ldots, 110\}$. The color gradient varies between dark gray for small w_C to light gray for higher values.



(a) Cedent's part vs. $x \cdot (1 - x^{qs})$ and x^{SL} (b) Reinsurer's part vs. $x \cdot x^{qs}$ and $x - x^{SL}$

Figure 2: Cedent's and reinsurer's part for different levels of w_C

The numerical solution in this setting leads to a reinsurance treaty that is a combination of linear and constant parts, as one can in Figure 2(a), for small levels of initial capital w_C. Increasing the amount of w_C leads to reinsurance contracts that tend toward the Quota-Share contract (dashed black line).

4. CONCLUSION

We studied optimal reinsurance contracts that involve both the cedent's and the reinsurer's perspective. In a first step we analyzed the linear combination of the utility functions of the two parties and derived a non-linear differential equation for the optimal retention function $h(x)$. We further observed that a risk-neutral reinsurer does not change the type of treaty of the "cedent-only" situation, if the expected value principle is applied, but the retention is changed. If both parties have exponential utility functions and the variance principle is applied, the parameters have a strong influence on the form of the optimal retention function. The optimal reinsurance contract numerically then turns out to be a combination of constant and linear retention functions.

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CROSS-GENERATIONAL COMPARISON OF STOCHASTIC MORTALITY OF COUPLED LIVES

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1. INTRODUCTION

In modeling mortality of coupled lives, it is essential to allow for dependence between the two remaining lifetimes. However, it is also vital to evaluate the change in mortality over time by comparing generations, both in terms of the change in individual mortality, and the change in dependence between the two lives. This should help life offices and pension schemes in their long-term planning. In this paper, we perform the previous task with reference to three generations whose males were born between 1900 and 1927 (1903 and 1930 for females). We proceed as follows. We model marginal survival functions of males and females using the doubly stochastic or Cox approach (with a stochastic mortality intensity). We incorporate dependence through an Archimedean copula model. We apply the marginal and copula model to a dataset of couples from a large North American insurer, widely used in the joint-life empirical literature (Frees et al. (1996)). We use marginal calibrations obtained via Maximum Likelihood and estimate the copula parameters through the Genest and Rivest method after having isolated the subset of complete data in the sample. As part of our derivation, we show that the censored-data methodology which one should in principle use for datasets like the one at hand, is misleading in generation comparisons. This is part of the theoretical contribution of the paper. We also show that not only the strength, as measured by Kendall's tau, but also the type of dependence (the copula) changes over generations. Dependence between males and females decreases from 44% to 28%, copulas go from Frank to Special. This complements the theoretical contribution of the paper. Last but not least, we study the pricing implications for a whole-life, joint-life and survivor annuity. This is our empirical contribution.

2. MARGINAL SURVIVAL FUNCTIONS

We adopt the doubly stochastic approach, based on a counting process, the intensity of which is a stochastic process (see for instance Cairns et al. (2006)). Under some technical conditions, marginal survival functions can be written as:

$$S_i^j(t) = \mathbb{E}[\exp(-\int_0^t \Lambda_i^j(s)ds)]$$

where i is the age of the life and j indicates the gender. The intensity Λ is assumed to evolve according to an affine process, so that the survival function can be obtained in closed form. On top of the affine assumption, we adopt an intensity belonging to the Feller family:

$$d\Lambda_i^j(s) = a_i^j \Lambda_i^j(s)ds + \sigma_i^j \sqrt{\Lambda_i^j(s)} dW_i^j(s),$$

where W_i^j is a one-dimensional Wiener process, $a_i^j > 0$, $\sigma_i^j \geq 0$. The choice is justified by the good fit of this intensity both in general – see Luciano and Vigna (2008) – and on the dataset to be used later – see Luciano et al. (2008). This leads to the following survival functions:

$$S_i^j(t) = \exp\left[\frac{1-\exp(b_i^j t)}{c_i^j + d_i^j \exp(b_i^j t)} \Lambda_i^j(0)\right]$$

with

$$\begin{cases} b_i^j = -\sqrt{\left(a_i^j\right)^2 + 2\left(\sigma_i^j\right)^2} \\ c_i^j = \dfrac{b_i^j + a_i^j}{2} \\ d_i^j = c_i^j - a_i^j \end{cases} .$$

3. COPULA MODEL

As customary in actuarial applications, we consider the one-parameter Archimedean class , whose expression is:

$$C(v,z) = \phi^{-1}\left(\phi(v) + \phi(z)\right)$$

where ϕ is the so-called generator, endowed with the following properties: $\phi : [0,1] \to [0,+\infty]$ and $\phi(1) = 0$. More specifically, in the calibration part we consider the following copulas:

- Clayton,
- Gumbel-Hougaard
- Frank
- 4.2.20 in Nelsen (2006), whose generator is $\phi(v) = \exp(v^{-\theta}) - e$
- Special, with generator $\phi(v) = v^{-\theta} - v^{\theta}$.

4. FITTING THE MARGINS AND COPULAS TO DATA

We choose three generations to be compared:

- a so-called Old Generation (male born between 1900 and 1913; female born between 1903 and 1916);
- a Middle Generation (m.b. 1907-20; f.b. 1910-23);
- a Young Generation (m.b. 1914-27; f.b. 1917-30).

Since the observation window in the original dataset comprises only 5 years, the data are censored from the right. Using all the (censored) data would mean to employ the method of Wang and Wells (2000). We argue that this would lead to misleading conclusions about generation-related dependence. Therefore, for copula fitting, we consider complete data only. The disadvantage is a significant reduction in the size of the data. The advantages are that

- we can use the familiar method of Genest and Rivest (1993) to estimate the copula parameter;
- this enables us to carry out a goodness-of-fit testing using the parametric bootstrap procedure of Genest et al. (2006).

We obtain the following parameters for marginal survival functions:

	OG Male	OG Female	MG Male	MG Female	YG Male	YG Female
a	961.045	790.232	810.051	1249.792	528.581	619.733
σ	0.007	0.057	2.426	0.021	0.019	0.5

As a consequence, the conditions for the intensity to stay positive are respected.
The estimates of Kendall's tau are: 0.4396 for the old generation, 0.3826 for the middle generation, 0.2792 for the young generation. So dependence decreases over generations, as one would intuitively expect. The copula parameters are estimated through the inversion of Kendall's tau approach of Genest and Rivest. This means to compute the distance between the theoretical K-function and the empirical K-function, according to three different norms: quadratic distance, Cramer-Mises distance and Kolmogorov-Smirnov distance. In all three cases, the best copula – i.e. the one with highest p-value – is also the one which minimizes all the distances: it turns out to be the Frank for the old generation, the Clayton for the middle, the Special for the young. The non-persistence of the same best-fit copula over generations should not come as a surprise, since all of them are Archimedean, but they differ in terms of upper tail dependence, and behaviour of dependence as the members of the couple age. What the sample seems to tell us is that - over generations - there is not only a decrease in dependence across members of the couple, but also a change in the type of dependence: for the old generation it is best represented by the Frank (no tail dependence, dependence decreasing while aging), for the middle by the Clayton (upper tail dependence, dependence constant while aging), for the young by the Special (upper tail dependence, dependence increasing while aging). However, while the decrease in dependence is strong (Kendall's tau going from almost 44% to 28%), the dominance of a given copula is less pronounced (the distances and p-values are not so strongly different). Apart from the statistical difference in dependence, we want to appreciate the impact on joint-life and survivor contracts.

5. ACTUARIAL APPLICATION

Let us consider a whole-life joint-life-and-survivor annuity contract, for which a benefit of 1 p.a. is payable in arrears while both lives are alive. The benefit reduces to $0 \leq R \leq 1$ on the first death, and continues until the last survivor dies.
The tables below show the net single premiums for the calibrated copula models and compare them to the ones obtained assuming independence of the two lives. They also give the ratio of the two premiums. Interest is at 2% p.a.. As expected, the ratio is decreasing for R increasing. Note the special cases of $R = 0$, which corresponds to a joint life annuity ; $R = 1$, which is the last survivor annuity ; and $R = 0.5$, for which dependence has no impact.

R	Frank	Independence	Ratio
0	8.722	7.72	1.13
$\frac{1}{4}$	10.273	9.772	1.051
$\frac{1}{3}$	10.79	10.456	1.032
$\frac{1}{2}$	11.823	11.823	1
$\frac{2}{3}$	12.857	13.191	0.975
$\frac{3}{4}$	13.374	13.875	0.964
1	14.924	15.926	0.937

R	Clayton	Independence	Ratio
0	12.326	11.261	1.095
$\frac{1}{4}$	13.754	13.222	1.04
$\frac{1}{3}$	14.23	13.875	1.026
$\frac{1}{2}$	15.183	15.183	1
$\frac{2}{3}$	16.135	16.49	0.978
$\frac{3}{4}$	16.611	17.143	0.969
1	18.039	19.104	0.944

R	Special	Independence	Ratio
0	17.547	16.41	1.069
$\frac{1}{4}$	19.635	19.066	1.03
$\frac{1}{3}$	20.33	19.951	1.019
$\frac{1}{2}$	21.722	21.722	1
$\frac{2}{3}$	23.113	23.492	0.984
$\frac{3}{4}$	23.809	24.378	0.977
1	25.896	27.034	0.958

The actuarial application to pricing of joint-life products and reversionary annuities shows then that not only we should dismiss the simplifying independence assumption, but we should also select different dependence parameters and copulas for different generations. Neglecting such differences has a non-negligible impact on the fair prices of annuities (unless $R = 0.5$).

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APPLYING CREDIT RISK TECHNIQUES TO EVALUATE THE ADEQUACY OF DEPOSIT GUARANTEE SCHEMES' FUND

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Deposit Guarantee Schemes (DGSs) are financial institutions whose main aim is to provide a safety net for depositors so that, if a credit institution fails, they will be able to recover their bank deposits up to a certain limit. The recent global financial crisis brought DGS at the centre of the political and financial debate. In July, 2010, the European Commission adopted a legislative proposal for an in-depth revision of the Directive on DGS, which aims at harmonizing and simplifying the schemes' functioning. We propose to investigate some implications of the proposal, focusing in particular on the DGS financing mechanisms, by simulating the DGS loss distribution using the Gaussian one-factor model. The DGS is thus treated as a portfolio of banks whose default probabilities are estimated from CDS spreads. The proposed approach is applied to a sample of Italian banks.

1. INTRODUCTION

Deposit Guarantee Schemes are financial institutions whose main aim is to reimburse depositors whenever their bank goes into default. If a credit institution fails, DGS intervenes and pays back the bank deposits up a certain amount. It is quite well-known that the existence of these institutions leads to some benefits: from depositors' point of view, DGS protects a part of their wealth from bank failures and avoid bank runs; from banking stability perspective, DGS contributes to strengthen the confidence in the financial sector, thus preventing bank runs, and to create a level playing field, thus avoiding competitive distortions.

These schemes are in place in many countries all over the world, like in the US, Canada, Russia, and Australia (Laeven (2002)). In this work we focus on schemes in place in Europe. In the European Union, Directive 94/19/EC (European Parliament and Council (1994)) obliged Member States to ensure the existence of at least one or more schemes on their territory, but required only minimum harmonization of rules across DGS; the Directive left a large degree of discretion to the schemes, especially in relation to the financing mechanisms (see Cariboni et al. (2008) and Cariboni et al. (2010)).

The 2008 global financial crisis brought DGS at the centre of the political and financial debate. This crisis emphasized the necessity of an in-depth revision of the whole Directive on DGS. As a result, in July 2010, the Commission adopted a legislative proposal on DGS (European Commission (2010)), which aims at simplifying and harmonizing many aspects of the DGS functioning left to the discretion of DGS up to now. The aspects mentioned in the proposal which will be more relevant for our work are the following.

- Simplification and harmonization of the scope of coverage. Only deposits by customers and by non financial corporations should be eligible for protection in all DGS.
- Harmonization of the financing mechanisms of DGS. All DGS should move to an ex-ante financing system, where financial resources are collected from member banks in advance on a regular basis.
- Choice of the target level for DGS funds. The target level for DGS' funds would be fixed equal to 2% of the amount of deposits eligible for protection. The transition period to let DGS reach the target level would be equal to 10 years.

In this work we want to investigate the adequacy of the features previously mentioned, especially focusing on the loss distribution of the DGS fund. Following a well recognized approach (Bennett (2002), Kuritzkes et al. (2002), and Sironi and Zazzara (2004)), DGS funds can be regarded of as portfolios of counterparty risks. These portfolios consist of individual exposures to insured banks, each of which has a small but non-zero probability of cause a loss to the fund. We simulate the empirical loss distribution of the DGS and we use it to investigate whether the target size of the fund fixed in the proposal is adequate to face potential banks' failures. The approach is applied to a sample of 52 Italian banks, accounting for around 60% of the total amount of eligible deposits in 2006.

2. EMPIRICAL DGS LOSS DISTRIBUTION

In order to obtain the DGS loss distribution, we first estimate banks' default probabilities and then we simulate banks' losses. Individual bank losses are aggregated to estimate the total loss hitting the fund. Banks' default probabilities are estimated from CDS and from risk indicators.

Credit Default Swaps are over the counter bilateral agreements where the protection buyer transfers the credit risk of a reference entity to the protection seller for a determined amount of time T (for a detailed description refer to Duffie and Singleton (2003) and to Schoutens and Cariboni (2009)). In this work we estimate banks default probabilities from the corresponding CDS spreads market data. In fact the CDS premia are among the best measures of the market pricing of credit risk currently available, due to standardized contract designs and the relatively high liquidity in the market (Raunig and Scheicher (2009)). Unfortunately, CDS contracts are written only on a very limited number of banks: in 2006, our reference year, CDS contracts were written on only around 40 European banks, and on only 4 Italian banks. In order to enlarge our sample, we make use of the entire set of European banks underlying a CDS contract to investigate possible relations between default probabilities and a set of financial indicators; this relation is then applied to the

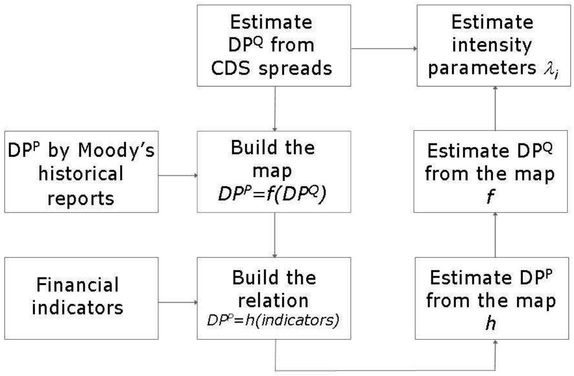

Figure 1: Procedure for default probabilities' estimation

sample of Italian banks to get an estimate of the default probability also for those institutions which do not have a CDS contract. In developing this approach, particular attention should be paid to the differences between the *risk-neutral* and the *historical* default probabilities (labeled DP^Q and DP^P, respectively). In our model, the default probabilities estimated from CDS spreads are risk-neutral because they come from an underlying pricing model. However, the correct probability measure to be associated to risk indicators is the historical probability, because it is backward looking like the financial indicators, which are built from balance sheet data. We thus have to find a proper map that allows us to move from one probability measure to the other. Figure 1 summarizes the procedure to estimate the default probabilities.

After DP^Q have been estimated from CDS spreads (see Schoutens and Cariboni (2009)), we build a map f between risk-neutral and historical default probabilities; we achieve this goal by associating every Moody's rating class with both an historical and a risk-neutral default probability[1] and we calibrate the quadratic and convex function that best fits the figures, according to the root mean square error criterium (see Schoutens (2003)).

We then study linear models $DP^P = \mathbb{X}\beta + \epsilon$, that link the historical default probabilities to risk indicators $\mathbb{X}$. In literature there exists a number of possible financial indicators (see, for example, Chan-Lau (2006)); in this work we restricted our attention to the risk indicators mentioned in the proposal[2]. Among all possible choices of indicators, the set of indicators that best explains the DP^P is the one listed in Table 1. By applying the linear model above studied we get an estimate of the DP^P for the sample of Italian banks, even for those who are not underlying a CDS contract; by applying the inverse of function f to the estimated DP^P we finally get an estimate of the risk-neutral default probabilities. We assume in our model that the default time of the i-th bank τ_i is exponentially distributed with intensity parameter λ_i, thus the term structure of the cumulative risk-neutral default probability up to time t for the i-th bank, $p_i(t)$, has the expression given by

[1]Data source: historical probabilities are estimated from default rates published by Emery et al. (2008), risk neutral probabilities from CDS spreads downloaded by Bloomberg, accessed from Bocconi University, Milan, 19th November 2010.

[2]In the proposal, financial indicators are the basis for the computation of risk-based contributions. Data source: Bankscope, accessed from Bocconi University, Milan, 19th November 2010.

ROAA	$\frac{\text{Exc. Capital}}{\text{Total Assets}}$
$\frac{\text{Liquid Assets}}{\text{Customer \& ST Funding}}$	$\frac{\text{Exc. Capital}}{\text{Risk-weighted Assets}}$
$\frac{\text{Net Loans}}{\text{Customer \& ST Funding}}$	$\frac{\text{Loan Loss Provisions}}{\text{Net Interest Revenue}}$
Cost to Income	$\frac{\text{Loan Loss Provisions}}{\text{Operating Income}}$

Table 1: Financial indicators

Equation (1):

$$p_i(t) = 1 - e^{-\lambda_i t}. \tag{1}$$

From risk-neutral default probabilities the default intensity parameters λ_i can be estimated accordingly.

We can now turn to the loss distribution. In order to build the empirical loss distribution of the scheme's fund, we must define what we mean by default. In the specific case of a DGS, a loss occurs if an insured bank fails, thus triggering a fund's payout. Following a common approach, a single bank is assumed to default when its asset value falls below a certain threshold. Banks' asset values $A_i(t)$ are modeled by a Gaussian one-factor model, following Vasicek (2002) approach:

$$A_i(t) = \sqrt{\rho} Y + \sqrt{1-\rho} X_i, \quad i = 1, \cdots, M. \tag{2}$$

It can be easily shown that the default time τ_i, corresponding to a draw A_i of the asset value process, satisfies Equation (3):

$$\tau_i = p_i^{(-1)} \left(\Phi(A_i)\right) = -\frac{\ln(1-\Phi(A_i))}{\lambda_i}. \tag{3}$$

In this study a bank i is regarded of as defaulting if the corresponding default time τ_i is lower than the transition period, i.e. 10 years. The corresponding loss is equal to the amount of covered deposits held by the failed bank. The total loss hitting the fund is estimated by aggregating individual banks' losses.

Default probabilities are estimated by using 2006 daily CDS spreads of 40 European banks; the sample of Italian banks representing the fund is made up of 52 banks, accounting for around 60% of the total amount of eligible deposits in 2006. The empirical loss distribution is built running 100000 Monte Carlo iterations, assuming common correlation factor $\rho_i = 70\%$ and recovery rates $R_i = 40\%$, equal for all banks. Table 2 reports the loss distribution of the whole sample of Italian banks: the figures show that the probability that the sample do not suffer any loss, within 10 years, is around 77.5%.

Table 3 reports the empirical loss distribution of the DGS fund. At the end of 10 years, the DGS is assumed to have set aside a fund equal to 2% of the amount of eligible deposits, which corresponds to around 7.72 billion €; by comparing this figure with the loss distribution reported in Table 2 we can see that this amount can cover around 90% of the banking system's losses.

Percentile	77.54	77.55	90	90.55	90.56	95	99	99.9	99.99	100
Loss (m€)	0	5	6,864	7,705	7,728	25,251	82,972	150,297	167,510	167,594

Table 2: Banking system's loss distribution. Data source: Eurostat and survey distributed by European Commission JRC among European DGS in 2009

Percentile	90	90.55	90.56	95	99	99.9	99.99	100
Loss (m€)	0	0	7	17,530	75,252	142,577	159,790	159,874

Table 3: DGS fund's loss distribution. Data source: Eurostat and survey distributed by European Commission JRC among European DGS in 2009

We further investigate the "optimal" size of the fund in order to have losses covered by the DGS in 95% of the cases. According to Figure 2, if we want the fund to cover losses up to the 95^{th} percentile of the distribution, the target size should be raised to around 6% of eligible deposits.

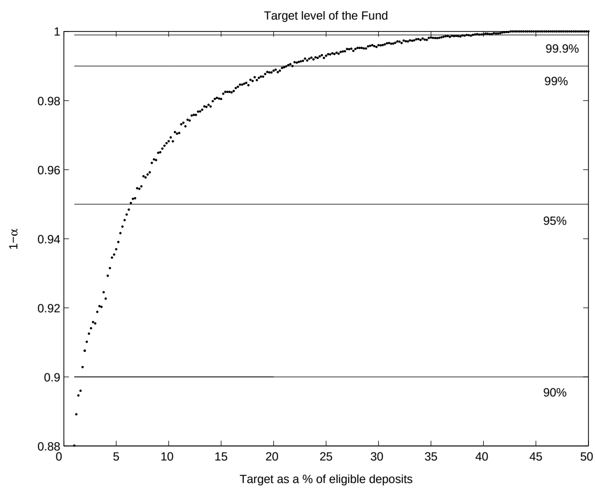

Figure 2: Percentage of losses covered by the fund when the target size ranges from 1% to 50% of eligible deposits

3. CONCLUSIONS

At the light of the recent financial crisis, the European Commission has adopted a legislative proposal for an in-depth revision of the Directive on Deposit Guarantee Schemes, with the aim of harmonizing the schemes' financing mechanisms and functioning. According to the proposal, DGS would have to reach a target for their funds equal to 2% of eligible deposits; the target would need to be reached within a transition period equal to 10 years. Focusing on a sample of Italian banks as of 2006, we have found out that such a designed fund can cover around 90% of banks' defaults during the transition period; if we want the fund to cover losses up to 95^{th} percentile of the distribution, the target size should be raised to around 6% of the amount of eligible deposits.

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A BAYESIAN COPULA MODEL FOR STOCHASTIC CLAIMS RESERVING

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1. INTRODUCTION

The estimation of Outstanding Loss Liabilities (OLLs) is crucial to reserve risk evaluation in risk management. Classical methods based on run-off triangles need a small amount of input data to be used. This fact determined their fortune, making them immediate to use, requiring the knowledge of the triangle of annual paid claims amount only. However, this fact constitutes also an important shortcoming, since using a small sample of data to predict future outcomes may possibly lead to inaccurate estimates. Anyway, their widespread use in professional practice encourages further improvements to limit this problem.

Starting from the beginning of this century, bayesian methods in estimating run-off triangles gained increasing attention as a tool to include expert judgement in stochastic models[1] and enlarge the information set on which reserves are computed. The use of Bayesian methods in loss reserving started decades ago, but it was the possibility of using Markov chain Monte Carlo (MCMC) fast computer-running algorithms that gave high flexibility to the application of this methodology, allowing for almost unrestricted distributional assumptions. De Alba (2002), De Alba and Nieto-Barajas (2008) – who introduced correlation among different accident years – and Ntzoufras and Dellaportas (2002) offer examples of how Bayesian methods can be implemented in the estimation of outstanding claims for a line of business, introducing prior information on both future claim amount (ultimate costs) and frequency. Simultaneously, some works tried to introduce the use of copulas – which gained increasing popularity in the finance world in the last decade – also in loss reserving[2].

The question of how to cope with dependent risks such as the losses an insurance company has to face in its different lines of business (LoBs) is surely of utmost importance. Current practice and Solvency II standard formulas account for diversification by means of linear correlation matrices

[1] For a comprehensive treatment on the use of copulas to aggregate expert opinions, see for example the seminal work Jouini and Clemen (1996).

[2] Copulas have been recently used in individual claim models (Zhao and Zhou (2010)).

estimated on a market-wide basis. Obviously, these correlation matrices can fail to capture the specificities insurance companies can present, due to geographical reasons or management choices. A few papers studied the application of copulas to run-off triangles estimation. Tang and Valdez (2005) used simulated loss ratios to aggregate losses from different LoBs. Li (2006) compared aggregation through the use of different copula functions, given distributional assumptions on the marginals. More recently, De Jong (2009) introduced a Gaussian copula model to describe dependence between LoBs.

This paper aims at combining both these two important aspects: Bayesian methods and the use of copulas. The Bayesian approach, introducing data coming from expert judgement, allows to include additional reliable information when estimating reserves and to derive full predictive distibutions. Copulas allow to obtain joint distributions in an easily tractable way, separating the process of defining the marginals and the dependence structure. Hence, we introduce prior information on the dependence structure, using Bayesian copulas in the aggregation of losses across LoBs. Up to our knowledge, this paper is the first attempt in introducing Bayesian copulas in stochastic claims reserving. Dalla Valle (2009) applied a similar technique to the problem of the estimation of operational risks. We adapt their approach to the aggregation of OLLs from different LoBs.

Combining a Bayesian approach to derive the marginal distributions of OLLs for each single LoB and the use of Bayesian copulas to aggregate them, it is possible to obtain a fully Bayesian model that incorporates expert judgement on the ultimate costs and development pattern of each LoB as well as on the dependence structure between them.

We apply this model to four lines of business of an Italian insurance company. We compare results obtained from the Bayesian copula model with those obtained from a standard copula approach, providing then a multi-dimensional application of the use of copula functions.

2. OUTLINE OF THE PAPER

This section briefly reviews the content of the paper. For a full account of what follows please refer to L. Regis, 2011, “A Bayesian copula model for stochastic claims reserving”, in “Three Essays in Finance and Actuarial Sciences”, Ph.D. Thesis.

We first present a standard Bayesian method to compute reserves for a single line of business, which we also use in the application to derive the marginal distributions. We consider an over-dispersed Poission model (ODP), following Merz and Wuthrich (2008):

$$
\begin{aligned}
&\mathbb{E}\left[\frac{X_{ij}}{\phi_i} \mid \Theta\right] = \frac{\mu_i\gamma_j}{\phi_i},\\
&\mathbb{V}\text{ar}\left[\frac{X_{ij}}{\phi_i} \mid \Theta\right] = \frac{\mu_i\gamma_j}{\phi_i},\\
&\text{with } \phi_i > 0, \mu_i > 0 \ \forall \ i = 1, \dots, I, \quad \text{(accident year)}\\
&\text{with } \gamma_j > 0 \ \forall \ j = 1, \dots, J, \quad \text{(development year)}\\
&\Theta = (\mu_1, \dots, \mu_I, \gamma_1, \dots, \gamma_J, \phi_1, \dots, \phi_I).
\end{aligned}
$$

In the application, the model has independent gamma priors on both μ’s and γ’s and it is renormalized fixing $\mu_1 = 1$. The over-dispersion parameter ϕ is set constant across accident years and it is derived from Pearson's residuals. A standard Metropolis-Hastings MCMC algorithm permits to find the predictive distribution of the OLLs for the considered line of business. The combination of such a Bayesian model for the estimation of the margins and of the Bayesian copula model we provide for the aggregation results in a fully Bayesian framework for the estimation of the overall reserves of a multi-line insurance company.

Thereafter, we tackle the problem of how to capture the dependence between LoBs. Correctly capturing the presence of dependence between the losses in different LoBs is intuitively a desirable feature of a good model for claims reserving. In Table 1 and 2, we compare the correlation matrix between the LoBs of an Italian insurance company, estimated from a time series of loss ratios, and the one the CEIOPS mandated to use when calculating reserve risk with the standard formula in the Quantitative Impact Studies (QIS)[3].

LoB				
	MTPL	**MOC**	**FP**	**TPL**
MTPL	1 (0)	0.4751 (0.0463)	0.4598 (0.0549)	0.5168 (0.0281)
MOC	0.4751 (0.0463)	1 (0)	0.8789 (0.000001)	0.7331 (0.0005)
FP	0.4598 (0.0549)	0.8789 (0.000001)	1 (0)	0.8748 (0.000002)
TPL	0.5168 (0.0281)	0.7331 (0.0005)	0.8747 (0.000002)	1 (0)

Table 1: Linear correlation between LoBs. The brackets report p-values.

LoB				
	MTPL	**MOC**	**FP**	**TPL**
MTPL	1	0.25	0.25	0.5
MOC	0.25	1	0.5	0.25
FP	0.25	0.5	1	0.25
TPL	0.5	0.25	0.25	1

Table 2: Correlation matrix estimated by CEIOPS and imposed to the participants to the Quantitative Impact Studies (QIS) to use in the evaluation of reserves, see European Commission (2010), p.203.

Table 1 and 2 clearly show that the industry-wide estimate proposed by CEIOPS and industry-specific ones can differ. Results on the correlation of a time series of realized losses, which we use in the application of our model further support this evidence.

Copula functions permit us - as we briefly review in the paper - to separate the process of estimating the marginal distributions $F(L_1), \ldots, F(L_n)$ of the OLLs of each LoB from the estimation

[3] The abbreviations in Table 1 and 2 stand for Motor Third Party Liability (MTPL), Motor Other Classes (MOC), Fire and Property (FP) and Third Party Liability (TPL).

of the dependence structure. Moreover, the latter can be modeled in a highly flexible way, since many copula functions are available to describe it and capture its (also non-linear) properties.

We then outline a simple procedure to obtain a joint distribution of OLLs for an n-dimensional non-life insurance company through the use of copulas:

1. derive the marginal distributions of the OLLs $F(L_1), \ldots, F(L_n)$ for each LoB independently. For this task, it is possible to resort to classical methods, simulation, as well as to the Bayesian technique we outlined above;

2. estimate the dependence structure between the L_i's for $i = 1, \ldots, n$;

3. choose a convenient copula function and evaluate its parameter(s). The copula will satisfy the uniqueness properties stated in Sklar's theorem (see e.g. Nelsen (2006)), depending on the form of its marginals.

Sampling from any n-dimensional copula obtained can be done exploiting the properties of conditional distributions. Then, we can easily evaluate the quantities of interest – such as relevant percentiles – on the simulated sample. Difficulties in the procedure lie mainly on the correct estimation of the dependence structure, which is a complicated task given the low (annual) frequency of the input data used in stochastic claims reserving models based on run-off triangles. The same observation applies to the choice of the most appropriate copula function. We compare the results of evaluating the OLLs of a multi-line insurer under different copula assumptions. Including company-specific measures of dependence in reserves' estimation together with industry-wide estimates as expert judgement could help in improving the quality of the predictions of future losses. Hence, we present a Bayesian approach to the use of copulas, by adding uncertainty on the parameters of the copula function.

The procedure – in general – works as follows:

1. choose a convenient distributional assumption for the prior of the copula parameter(s) $\boldsymbol{\theta}$, $\pi(\boldsymbol{\theta})$

2. compute, using Bayes' theorem, the posterior distribution of the parameter given the input data:
$$f(\boldsymbol{\theta} \mid \mathbf{x}) = f(\mathbf{x} \mid \boldsymbol{\theta})\pi(\boldsymbol{\theta}),$$
where $\mathbf{x}$ denotes the $n \times T$ matrix of observations (T is the number of observations).

A convenient choice of the prior distribution involves the choice of priors whose densities are conjugate to the one of the distribution of the estimation object – in our context, OLLs per accident year. In the paper, we provide a detailed description of the procedure for a Bayesian Gaussian copula. In that case, we choose an Inverse Wishart prior distribution for the covariance matrix, which is conjugate to the multivariate Gaussian. Hence, the posterior distribution for the covariance matrix is again an Inverse Wishart with parameters that can be estimated from the data.

Finally, we apply the framework to an Italian insurance company. We first compare the distributions of overall OLLs as obtained through standard copulas, under different copula type choices (Gaussian, t and Archimedean copulas) when correlation is estimated through a time-series of loss ratios and when the QIS 5 matrix of Table 2 is used. Then, we obtain the simulated distributions resulting from our Bayesian Gaussian copula model. Parameters of the prior distribution are set conveniently to match its mean to the QIS 5 correlation matrix.

Further extensions involve the application of the Bayesian framework to a t-copula and the introduction of an hierarchical model for the estimation of parameters.

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MODELLING CLAIM COUNTS OF HOMOGENEOUS INSURANCE RISK GROUPS USING COPULAS

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1. INTRODUCTION

We study an application of copula modelling to insurance using count data of the automobile line of business. We consider the following homogeneous risk groups: third party liability property damages, third party liability bodily injury and material own damages. First, we model the marginal behaviour of each group, then we estimate a tri-variate copula. Data suggests similar correlations between groups. We did a continuous and a discrete approach. Due to the limitations of a direct discrete approach we perform a continuous approximation. In the discrete case we fit a negative binomial to each risk group and in the continuous one we try the gamma and normal approximations.

As our application relies on the assumption of parametric univariate marginals we perform the goodness-of-fit tests by proposing a parametric extension of the test presented by Genest et al. (2009). The test is based on the empirical copula,

$$\widehat{C}(\mathbf{u}) = \frac{1}{t+1}\sum_{j=1}^{t} I\{Z_{j1} \leq u_1, ..., Z_{jn} \leq u_n\}$$

where $\mathbf{u} = (u_1, ..., u_n) \in [0,1]^n$ and I is the indicator function. In the semiparametric approach by Genest et al. (2009) the sample of the vector $Z$is given by

$$\mathbf{z}_j = (z_{j1}, ..., z_{jn}) = \left(\frac{\text{Rank}(x_{j1})}{n+1}, ..., \frac{\text{Rank}(x_{jn})}{n+1}\right) \quad j = 1, ..., t$$

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where Rank(x_{j1}) is the rank of x_{ji} amongst the sample of the claim counts $(x_{1i},...,x_{ni})$. To evaluate the goodness-of-fit of a parametric copula model we propose that the sample of the vector Z is given by

$$\mathbf{z}_j = (z_{j1},...,z_{jn}) = (F_1(x_{j1}),...,F_n(x_{jn})) \quad j = 1,...,t.$$

The test is based on a statistical test that compares $\widehat{C}(z)$ with an estimated $C_{\widehat{\theta}}(z)$, of the theoretical copula C_θ. This statistical test is given by,

$$\widehat{T} = t\int_{[0,1]^n} \left\{\widehat{C}(\mathbf{z}) - C_{\widehat{\theta}}(\mathbf{z})\right\}^2 d\widehat{C}(\mathbf{z}) = \sum_{j=1}^{t} \left\{\widehat{C}(\mathbf{z}_j) - C_{\widehat{\theta}}(\mathbf{z}_j)\right\}^2. \qquad (1)$$

A bootstrap procedure is required to compute the p-value of the test (1). The steps of the bootstrap technique are detailed in Berg (2009) and could be easily adapted to the parametric approach. For the univariate distributions we do the standard goodness-of-fit tests.

We work with a sample of the automobile portfolio of an insurer operating in Portugal. It has monthly observations of claim counts from 2000 to 2008. As expected, the estimated values of the Kendall's *tau* presented in Table 1 reveal dependence among the three groups. The descriptive statistics presented in Table 2 show some negative skewness for the material own damages risk group which maybe be due to the existence of a franchise and an upper capital limit (value of the vehicle). For details, please see Santos (2010).

Risk group	TPL property damages	TPL bodily injury	Material own damages
TPL property damages	1	0.437	0.492
TPL bodily injury		1	0.299
Material own damages			1

Table 1: Kendall's tau matrix of the claim counts

Risk group	TPL property damages	TPL bodily injury	Material own damages
Mean	4 028	337	1 198
Standard Deviation	568.93	54.07	121.46
Coefficient of Variation	14%	16%	10%
Skewness	0.13	0.38	−0.19
Kurtosis	−1.05	0.06	−0.61

Table 2: Descriptive statistics of the data

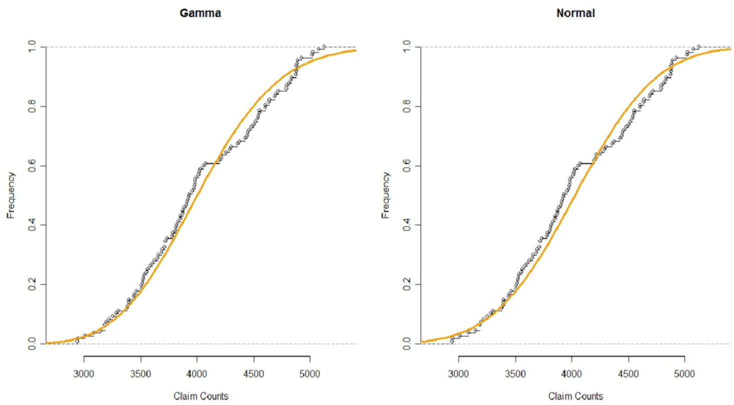

Figure 1: TPL property damages - Gamma and normal *vs* empirical distribution.

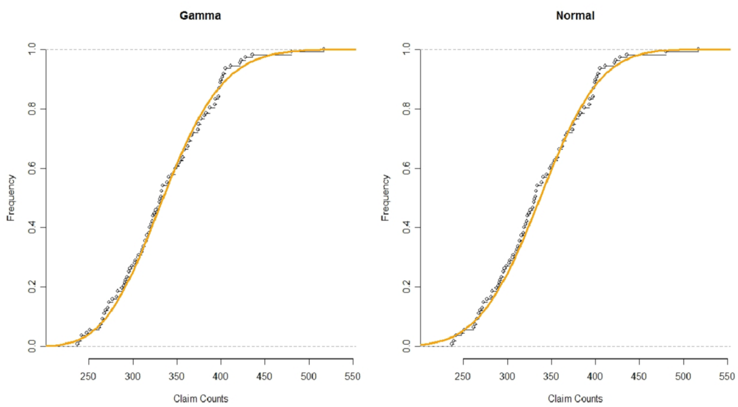

Figure 2: TPL bodily injury - Gamma and normal *vs* empirical distribution.

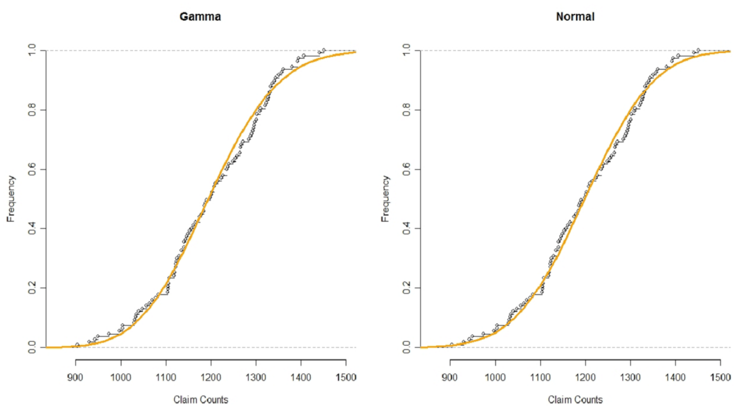

Figure 3: Material own damages - Gamma and normal *vs* empirical distribution.

2. CONTINUOUS MODELLING

We consider gamma and normal distributions approximations to model the marginal behaviour for the claim counts of each risk group. The former can be viewed as a *continuous version* of the negative binomial distribution, when the random variables do not take zero values and do not have a large number of repeated values. The latter is taken from the Central Limit Theorem.

Parameters estimation is carried out using the Inference For Margins method (IFM). It is a two-step method that first estimates the marginal parameters and then calibrates the copulae parameters. It does n separate optimizations of the univariate likelihoods, followed by an optimization of the multivariate likelihood as a function of the dependence parameter vector. Table 3 shows the maximum likelihood parameter estimates (ML) for the gamma and normal approximations as well as the p-values of the Kolmogorov-Smirnov tests for the three risk groups. Accordingly, Figures 1-3 show plots of the empirical and approximating distributions.

	Gamma distribution			Normal distribution		
Risk group	α	β	p-value	μ	σ	p-value
TPL property damages	50.39	0.013	43.94%	4028	568.93	40.37%
TPL bodily injury	39.60	0.117	96.60%	337	54.07	76.34%
Motor own damages	95.91	0.080	59.61%	1198	121.46	65.06%

Table 3: ML estimates of gamma and normal fits and p-values

We tried five different trivariate copula families and estimated their parameters considering both gamma and normal distributions. These results are shown in Table 4. We tried the t-copula but the estimated degrees of freedom were high and thus the t-copula comes close to the Gaussian copula. According to Embrechts et al. (2003) a robust estimator for the components of the correlation matrix R of the Gaussian copula is given by $R_{ij} = \sin(\pi\widehat{\tau_{ij}}/2)$.

	Gamma distribution			Normal distribution		
Copulas	θ_1	θ_2	p-value	θ_1	θ_2	p-value
Gumbel	1.5437	1.6242	0%	1.5072	1.5781	1.50%
Nelsen	1.0474	1.0923	28.47%	1.0597	1.1043	6.89%
Cook-Johnson	0.9482	-	10.09%	1.0648	-	5.19%
Gaussian	-	-	41.36%	-	-	39.86%

Table 4: Copulae parameters estimates with gamma and normal distributions and p-values

According to the goodness-of-fit test, with a significance level greater than 10%, we came out with the following models to fit the claim counts: Gaussian copula with normal marginal distributions; Nelsen's copula with gamma marginal distributions; Gaussian copula with gamma marginal distributions; Cook-Johnson's copula with gamma marginal distributions.

Cook-Johnson's copula assumes an equal level of association for all pairs of random variables which is a very restrictive property. For instance, according to the Kendall's *tau* in Table 1 we see that may not be true. Nelsen's copula with gamma marginal distributions allows skewness in the data whereas the Gaussian copula leads to a radial symmetric distribution. Since our data has some asymmetry, we conclude that Nelsen's copula with gamma distribution marginals should be preferred to model the joint claim counts of the three risk groups.

3. DISCRETE MODELLING

In the discrete modelling we consider a mixed Poisson distribution with a structure gamma distribution for the marginals, leading to a negative binomial distribution. Table 5 shows the maximum likelihood parameter estimates for the negative binomial distribution as well as the p-values of the Chi-squared test. Figure 4 shows the negative binomial fit *versus* empirical distribution.

	Negative binomial distribution		
Risk group	α	p	p-value
TPL property damages	50.77	0.012	14.91%
TPL bodily injury	43.92	0.115	24.81%
Motor own damages	105.96	0.081	78.11%

Table 5: ML parameter estimates of negative binomial distribution and p-values

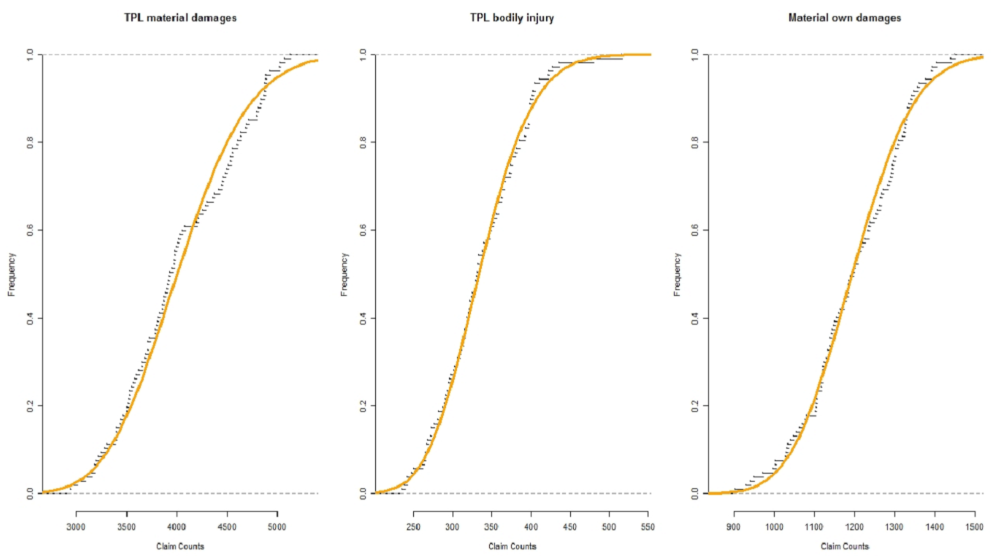

Figure 4: Negative binomial *vs* empirical distribution

The discrete approach should be the natural method to fit the claim counts, however it has limitations since the copula theory has serious restrictions when the marginals are discrete. Thus, we can neither properly estimate the copula and its parameters nor fully test the fit. To overcome these limitations we estimate copulae for the structure distributions. Since the parameters of the copulae represent the dependence parameters we estimate them using the claim counts sample. However, since the structure variables are not observable we cannot perform a goodness-of-fit test for the copulae. Tables 6 shows the estimates for the structure gamma distributions and Table 7 shows the parameters estimates for the copulae.

	Structure gamma	
Risk group	α	$\beta = p/(1-p)$
TPL property damages	50.77	0.0126
TPL bodily injury	43.92	0.1303
Motor own damages	105.96	0.0884

Table 6: Parameter estimates for the structure gamma distribution

Archimedean	Structure gamma	
Copulas	θ_1	θ_2
Gumbel	1.5011	1.5829
Nelsen	1.0195	1.0704
Cook-Johnson	0.8485	-

Table 7: Archimedean copulas parameter estimates for the structure gamma for marginals

Comparing the estimates obtained in the discrete modelling with the ones obtained in the continuous case we see that they are similar in the cases of Gumbel, Nelsen and Cook-Johnson's copulas.

4. CONCLUSION

According to the automobile data illustration presented, as multivariate model to fit the claim counts between TPL property damages, TPL bodily injury and material own damages risk groups, Nelsen's copula should be chosen with gamma marginals. The discrete approach presented seems to confirm this conclusion, and it is an interesting line of research for the future. Moreover, the results obtained for the degrees of freedom of the t-copula support the absence of a tail dependence in the risk groups. This seems reasonable because extreme events are not covered by these groups. See Santos (2010) for details.

We remark that this application has at least one limitation due to a possible existence of seasonality in the data. This is not captured by the IFM method that assumes independent observations.

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GENERAL MODEL FOR MEASURING THE UNCERTAINTY OF THE CLAIMS DEVELOPMENT RESULT (CDR)

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We consider the problem of claims reserving and estimating in the setup of run-off triangles. This problem is motivated by the need of monitoring the randomness of claim developments up to the time when the ultimate claim is finally settled. This aspect of claims reserving relies, typically, on a long-term point of view. This is in contrast with the short-term horizon inherent to models describing the total risk for an insurance company, such as the one-year risk perspective used in the Solvency II project. The challenge of bridging the gap between these two viewpoints gave rise to some innovative research in the study of reserving processes. One of the first papers dealing with the one-year reserve risk was the one of Merz and Wüthrich (2008). In the special case of a pure Chain-ladder estimate, they provided analytic formulae for the mean squared error of the predictions of the run-off result, referred to as the *claims development result* (CDR). Their methods rely on an extension of the well-known Mack (1993) model.
The present paper intends to provide a general methodology for measuring the uncertainty of CDR. Our approach largely extends that of Merz and Wüthrich (2008). We will make an instrumental use of the notation and methods of this paper and follow the arguments of the proof of their main results.

1. INTRODUCTION

Merz and Wüthrich defined in Merz and Wüthrich (2008) the *claims development result* (CDR) at time $I+1$ for accounting year $(I, I+1]$ as the difference between two successive predictions of the total ultimate claim. The first prediction is done at time I (with the available information up to time I), and the second one is made one period later at time $I+1$ (with the updated information available at time $I+1$). Merz and Wüthrich base their study of the prediction of CDR, and of the possible fluctuations around this prediction (prediction uncertainty) on a distribution-free Chain Ladder method.
In the present paper we extend their model to a more general class of models based on age-to-age factors. The Chain Ladder Model of Merz and Wüthrich (2008) turns out to be a particular case

of our general model. In the final section we apply our results to four methods (Chain Ladder included) of claims reserving often used by practitioners. The proofs of the main results are given in Sloma (2011).

2. NOTATION

- $C_{i,j}$ - cumulative payments for accident year $i \in \{0, \ldots, I\}$ until development year $j \in \{0, \ldots, J\}$
- $C_{i,j}$ - random variables observable for calendar years $i + j \leq J$ and non-observable (to be predicted) for calendar years $i + j > J + 1$
- $C_{i,J}$ - ultimate claim for accident year i
- We assume that $I = J$ (dataset as run-off triangle, see Table 1)

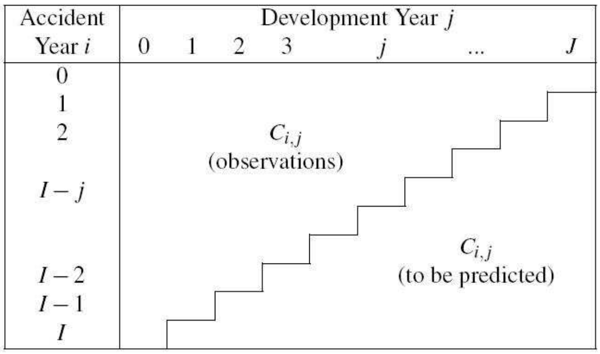

Table 1: Run-off triangle ($I = J$)

- $D_I = \{C_{i,j} : i + j \leq I; i \leq I\}$ - claims data available at time $t = I$
- $D_{I+1} = \{C_{i,j} : i + j \leq I + 1; i \leq I\}$ - claims data available at time $t = I + 1$

3. EXTENSION OF MACK'S MODEL FOR THE CHAIN LADDER METHOD

Define : $F_{i,k} = C_{i,k+1}/C_{i,k}$ - individual development factors, $\gamma_{i,k}$ - positive random variables $\sigma(C_{i,k})$-measurable and $\sigma(C_{i,k})$ - σ- field generated by $C_{i,k}$.

3.1. Model Assumptions

(M.1) The accident years $(C_{i,0}, \ldots, C_{i,J})_{0 \leq i \leq I}$ are independent.

(M.2) $(C_{i,j})_{0 \le j \le J}$ are Markov chains.

(M.3) There exist constants $f_k > 0$ and $\sigma_k^2 > 0$ such that for all $0 \le i \le I$ and $0 \le k \le J-1$ we have

$$\mathrm{E}(F_{i,k} \mid C_{i,0}, \ldots, C_{i,k}) = \mathrm{E}(F_{i,k} \mid C_{i,k}) = f_k$$
$$\mathrm{Var}(F_{i,k} \mid C_{i,0}, \ldots, C_{i,k}) = \mathrm{Var}(F_{i,k} \mid C_{i,k}) = \frac{\sigma_k^2}{\gamma_{i,k}} \quad .$$

3.2. Model Estimators

$$\widehat{f}_k^I = \frac{\sum_{i=0}^{I-k-1} \gamma_{i,k} F_{i,k}}{\sum_{i=0}^{I-k-1} \gamma_{i,k}}, \quad \text{and} \quad \widehat{f}_k^{I+1} = \frac{\sum_{i=0}^{I-k} \gamma_{i,k} F_{i,k}}{\sum_{i=0}^{I-k} \gamma_{i,k}}, \quad 0 \le k \le J-1,$$
$$\widehat{\sigma}_k^2 = \frac{1}{I-k-1} \sum_{i=0}^{I-k-1} \gamma_{i,k} (F_{i,k} - \widehat{f}_k)^2, \quad \text{for } 0 \le k \le J-2,$$
$$\widehat{\sigma}_{J-1}^2 = \min(\widehat{\sigma}_{J-2}^4 / \widehat{\sigma}_{J-3}^2, \min(\widehat{\sigma}_{J-3}^2, \widehat{\sigma}_{J-2}^2)).$$

4. CLAIMS DEVELOPMENT RESULT (CDR)

4.1. True CDR

- Single Accident Year

$$\mathrm{CDR}_i(I+1) = \mathrm{E}[C_{i,J} \mid D_I] - \mathrm{E}[C_{i,J} \mid D_{I+1}].$$

- Aggregation over Prior Accident Year

$$\mathrm{CDR}(I+1) = \sum_{i=1}^{I} \mathrm{CDR}_i(I+1).$$

4.2. Observable CDR

- Single Accident Year

$$\widehat{\mathrm{CDR}}_i(I+1) = \widehat{C}_{i,J}^I - \widehat{C}_{i,J}^{I+1},$$

where

$$\widehat{C}_{i,J}^I = C_{i,I-i} \prod_{j=I-i}^{J-1} \widehat{f}_j^I \quad \text{and} \quad \widehat{C}_{i,J}^{I+1} = C_{i,I-i+1} \prod_{j=I-i+1}^{J-1} \widehat{f}_j^{I+1}.$$

- Aggregation over Prior Accident Year

$$\widehat{\mathrm{CDR}}(I+1) = \sum_{i=1}^{I} \widehat{\mathrm{CDR}}_i(I+1).$$

5. MEAN SQUARE ERROR OF PREDICTION (MSEP) OF CDR

The conditional MSEP considered here gives the prospective solvency point of view and quantifies the prediction uncertainty in the budget value at 0 for the observable CDR at the end of the accounting period. In the solvency margin we need to hold risk capital for possible negative deviations of CDR from 0.

$$\mathrm{msep}_{\widehat{\mathrm{CDR}}_i(I+1)|D_I}(0) = \mathrm{E}\left[\left(\widehat{\mathrm{CDR}}_i(I+1) - 0\right)^2 \mid D_I\right],$$

$$\mathrm{msep}_{\sum_{i=1}^{I}\widehat{\mathrm{CDR}}_i(I+1)|D_I}(0) = \mathrm{E}\left[\left(\sum_{i=1}^{I}\widehat{\mathrm{CDR}}_i(I+1) - 0\right)^2 \mid D_I\right].$$

6. MAIN RESULTS

6.1. Single Accident Year

$$\widehat{\mathrm{msep}}_{\widehat{\mathrm{CDR}}_i(I+1)|D_I}(0) = \left(\widehat{C}^I_{i,J}\right)^2 \cdot \left(\widehat{\Gamma}^I_{i,J} + \widehat{\Delta}^I_{i,J}\right),$$

where

$$\widehat{\Gamma}^I_{i,J} = \frac{\widehat{\sigma}^2_{I-i}/\left(\widehat{f}^I_{I-i}\right)^2}{\gamma_{i,I-i}} + \sum_{j=I-i+1}^{J-1} \frac{\widehat{\sigma}^2_j/\left(\widehat{f}^I_j\right)^2}{\left(\beta^{I+1}_j\right)^2} \cdot \gamma_{I-j,j},$$

$$\widehat{\Delta}^I_{i,J} = \frac{\widehat{\sigma}^2_{I-i}/\left(\widehat{f}^I_{I-i}\right)^2}{\beta^I_{I-i}} + \sum_{j=I-i+1}^{J-1} \frac{\widehat{\sigma}^2_j/\left(\widehat{f}^I_j\right)^2}{\beta^I_j} \cdot \left(\frac{\gamma_{I-j,j}}{\beta^{I+1}_j}\right)^2,$$

$$\beta^I_j = \sum_{i=0}^{I-j-1} \gamma_{i,j} \quad \text{and} \quad \beta^{I+1}_j = \sum_{i=0}^{I-j} \gamma_{i,j}.$$

6.2. Aggregation over Prior Accident Year

$$\widehat{\text{msep}}_{\sum_{i=1}^{I}\widehat{\text{CDR}}_i(I+1)|D_I}(0) = \sum_{i=1}^{I} \widehat{\text{msep}}_{\widehat{CDR}_i(I+1)|D_I}(0) + 2\sum_{k>i>0} \widehat{C}_{i,J}^I \cdot \widehat{C}_{k,J}^I \left(\widehat{\Upsilon}_{i,J}^I + \widehat{\Lambda}_{i,J}^I\right),$$

where

$$\widehat{\Upsilon}_{i,J}^I = \frac{\widehat{\sigma}_{I-i}^2 / \left(\widehat{f}_{I-i}^I\right)^2}{\beta_{I-i}^{I+1}} + \sum_{j=I-i+1}^{J-1} \frac{\widehat{\sigma}_j^2 / \left(\widehat{f}_j^I\right)^2}{\left(\beta_j^{I+1}\right)^2} \cdot \gamma_{I-j,j},$$

$$\widehat{\Lambda}_{i,J}^I = \frac{\widehat{\sigma}_{I-i}^2 / \left(\widehat{f}_{I-i}^I\right)^2}{\beta_{I-i}^I} \cdot \left(\frac{\gamma_{i,I-i}}{\beta_{I-i}^{I+1}}\right) + \sum_{j=I-i+1}^{J-1} \frac{\widehat{\sigma}_j^2 / \left(\widehat{f}_j^I\right)^2}{\beta_j^I} \cdot \left(\frac{\gamma_{I-j,j}}{\beta_j^{I+1}}\right)^2.$$

7. NUMERICAL EXAMPLE

Following Mack (1999) we define, for $0 \le i \le I$ end $0 \le j \le I-i$,

$$\gamma_{i,j} = w_{i,j} \cdot C_{i,j}^{\alpha},$$

where $\alpha \ge 0$ and $w_{i,j} \in (0,1]$ are arbitrary weights which can be used by the actuary to down-weight any outlying $F_{i,j}$.
We choose the parameter α and the weights $w_{i,j}$ to obtain the following four methods (A-D) often used by practitioners

A. **Chain Ladder Model** ($\alpha = 1$, $w_{i,j} = 1$ for all i, j) (see Merz and Wüthrich (2008) and Mack (1999)),

$$\widehat{f}_k = \frac{\sum_{i=0}^{I-k-1} C_{i,k} F_{i,k}}{\sum_{i=0}^{I-k-1} C_{i,k}} = \frac{\sum_{i=0}^{I-k-1} C_{i,k+1}}{\sum_{i=0}^{I-k-1} C_{i,k}}, \quad \text{for } 0 \le k \le J-1.$$

B. **Mean Model** The estimators of f_k are the straightforward averages of the observed individual development factors $F_{i,j}$ ($\alpha = 0$, $w_{i,j} = 1$ for all i, j),

$$\widehat{f}_k = \frac{1}{I-k} \sum_{i=0}^{I-k-1} F_{i,k}, \quad \text{for } 0 \le k \le J-1.$$

C. **Linear Regression Model** The estimators of f_k are the results of an ordinary regression of $\{C_{i,k+1}\}_{i\in\{0,\ldots,I-k-1\}}$ against $\{C_{i,k}\}_{i\in\{0,\ldots,I-k\}}$ with intercept 0 ($\alpha = 2$, $w_{i,j} = 1$ for all i, j),

$$\widehat{f}_k = \frac{\sum_{i=0}^{I-k-1} C_{i,k}^2 F_{i,k}}{\sum_{i=0}^{I-k-1} C_{i,k}^2} = \frac{\sum_{i=0}^{I-k-1} C_{i,k} C_{i,k+1}}{\sum_{i=0}^{I-k-1} C_{i,k}^2}, \quad \text{for } 0 \le k \le J-1.$$

D. **Sample Median Model** ($\alpha = 0$, $w_{i,j} \in \{0;1\}$ for all i, j). The weights $w_{i,j}$ are chosen in the way that the estimators of f_k are given by

$$\widehat{f_k} \cong \text{median}\{F_{i,k} : i \in \{0,\ldots,I-k-1\}\}, \text{ where}$$

$$\text{median}\{X_i : i \in \{0,\ldots,n\}\} = \begin{cases} X_{(\frac{n+1}{2})} & \text{if } n \text{ is odd} \\ \dfrac{X_{(\frac{n}{2})} + X_{(\frac{n}{2}+1)}}{2} & \text{otherwise} \end{cases}$$

$X_{(k)}$ denotes the k^{th} order statistics of the sample $X_1,\ldots,X_n$.

8. NUMERICAL RESULTS

	$\widehat{msep}_{\widehat{CDR}_i(I+1)\|D_I}(0)^{1/2}$			
i	A	B	C	D
0				
1	567	563	572	0
2	1'488	1'501	1'475	1'474
3	3'923	3'863	3'982	450
4	9'723	9'634	9'812	472
5	28'443	28'320	28'563	473
6	20'954	20'460	21'475	4'614
7	28'119	27'485	28'783	1'578
8	53'320	52'017	54'690	1'904
Total	**81'080**	**79'749**	**82'468**	**8'771**
	A/A	B/A	C/A	D/A
Total (%)	**100,0%**	**98,4%**	**101,7%**	**10,8%**

Table 2: Estimates of MSEP (run-off triangle from Table 2 of Merz and Wüthrich (2008))

	$\widehat{msep}_{\widehat{CDR}_i(I+1)\|D_I}(0)^{1/2}$			
	A	B	C	D
Total	**33'470**	**143'670**	**14'462**	**3'787**
	A/A	B/A	C/A	D/A
Total (%)	**100,0%**	**429,2%**	**43,2%**	**11,3%**

Table 3: Estimates of MSEP (run-off triangle from Table 1 of England and Verrall (2002))

9. CONCLUSIONS

The methodology developed in Merz and Wüthrich (2008) is applied in practice within the Solvency II framework in the context of estimation of the one year volatility of reserves (see CEIOPS (2010), methods 4-6, p. 64-67).

Our general model gives an alternate approach for such applications. Numerical results from Table 2 (except for model D, the results are close to each other) and Table 3 (divergent results) show that the choice of model for reserving processes is still an open challenging problem and underlines the importance of statistical inference methods to properly assess the model structure in each case.

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De handelingen van deze contactfora vormen een aparte publicatiereeks van de Academie.

Contactforum "Actuarial and Financial Mathematics Conference" (10-11 februari 2011, Prof. M. Vanmaele)

Deze handelingen bevat een neerslag van bijdragen op de "Actuarial and Financial Mathematics Conference 2011". Dit contactforum was dit jaar aan zijn 9de editie toe en heeft zijn plaats veroverd tussen de internationale conferenties die focussen op de wisselwerking tussen financieel en actuarieel wiskundige technieken. Gespreid over de twee dagen kwamen verschillende experten vertellen over nieuwe modellen en technieken voor onder andere het modelleren van verschillende soorten risico en het construeren van een optimale investeringsportefeuille. Ook jonge onderzoekers kregen de kans hun onderzoeksresultaten ofwel in een voordracht ofwel via een poster aan een ruim publiek voor te stellen bestaande uit academici uit binnen- en buitenland alsook collega's uit de bank- en verzekeringswereld. In deze publicatie vindt u een samenvatting van een deel van deze presentaties.