KONINKLIJKE VLAAMSE ACADEMIE VAN BELGIE
VOOR WETENSCHAPPEN EN KUNSTEN

ACTUARIAL AND FINANCIAL MATHEMATICS CONFERENCE

Interplay between Finance and Insurance

February 9-10, 2012

**Michèle Vanmaele, Griselda Deelstra, Ann De Schepper,
Jan Dhaene, Wim Schoutens, Steven Vanduffel & David Vyncke (Eds.)**

CONTACTFORUM

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Handelingen van het contactforum "Actuarial and Financial Mathematics Conference. Interplay between Finance and Insurance" (9-10 februari 2012, hoofdaanvrager: Prof. M. Vanmaele, UGent) gesteund door de Koninklijke Vlaamse Academie van België voor Wetenschappen en Kunsten.

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Actuarial and Financial Mathematics Conference
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KONINKLIJKE VLAAMSE ACADEMIE VAN BELGIE
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Actuarial and Financial Mathematics Conference
Interplay between finance and insurance

PREFACE

The fifth edition of the "Actuarial and Financial Mathematics Conference" on February 9 and 10, 2012 was a great success. In the buildings of the Royal Flemish Academy of Belgium for Science and Arts in Brussels we welcomed 150 participants on both days coming from 17 different countries. This international conference has become an important meeting place for professionals of the banking and insurance business such as Dexia Bank, KBC Bank en Verzekeringen, Generali Belgium, Delta Lloyd, AG insurance, Risk Dynamics, Reacfin amongst others. We counted 60 of them. Some are former master or PhD students of the organising universities.

The successful format of this conference consists of 9 plenary 45 minutes talks by invited internationally renowned speakers, 8 contributed talks of 30 minutes selected by the scientific committee from the many submissions and a poster session during the coffee and lunch breaks.
The first day *Thaleia Zariphopoulou (University of Oxford, UK and University of Texas at Austin, USA), Rüdiger Frey (Universität Leipzig, Germany and WU Vienna, Austria), Cornelis W. Oosterlee (CWI Amsterdam & Delft University of Technology, The Netherlands), Erhan Bayraktar (University of Michigan, USA) and Alexandre Novikov (University of Technology of Sydney, Australia)* talked about their recent research results with a focus on finance. Their clear expositions gave the audience some insight in time consistency for mean-variance portfolio optimization, in structural credit risk models with incomplete information of the asset value, in the COS-method as an efficient pricing method for financial derivatives, in a framework for solving optimal liquidation problems in limit order books and in analytical approximations and numerical results for pricing of volume-weighted average options.
The second day, the attendants had the opportunity to listen to the following invited speakers: *Stéphane Loisel (Université Lyon 1 - I.S.F.A., France), Ludger Rüschendorf (University of Freiburg, Germany), Annamaria Olivieri (University of Parma, Italy) and Anja De Waegenaere (Tilburg University, The Netherlands).* All talks dealt with insurance issues but with an interplay with finance such as explicit formulas for ruin probabilities and related quantities in collective risk models with dependence among claim sizes and among claim inter-occurrence times, the description of possible influence of positive dependence on the magnitude of risk in a portfolio vector, the joint modelling of financial and mortality/longevity risks when modelling variable annuities, two potential strategies to mitigate the adverse effects of longevity risk on pension providers.

The eight talks of the contributed speakers selected by the scientific committee matched very well the invited talks they were programmed with. These talks with topics in finance and insurance were given by *Salvatore Federico (Université Paris 7, Franc), Asma Khedher (CMA, University of Oslo, Norway), Marie Chazal (Université Libre de Bruxelles, Belgium), Łukasz Delong (Warsaw School of Economics, Poland), Monika Forys (KU Leuven, Belgium), Roman Muraviev (ETH Zürich, Switzerland), Matthias Boerger (IFA Ulm & Ulm University, Germany) and Donatien Hainaut (ESC Rennes Business School and ENSAE, France).*
New this year was the poster storm session during which the 16 poster presenters got each one minute and two slides to tell the audience about the results presented on their poster. By these presentations more participants were attracked to the poster session.
We thank all speakers and presenters for their enthusiasm and their nice presentations which made the conference a great success.

The proceedings contain three articles related to the invited talks, three papers containing results presented in the contributed talks and eight short communications of poster presenters of the poster sessions on both conference days. In this way the reader gets an overview of the topics and activities at the conference.

We are much indebted to the members of the scientific committee, *Hansjörg Albrecher (HEC Lausanne, Switzerland), Freddy Delbaen (ETH Zurich, Switzerland), Michel Denuit (Université Catholique de Louvain, Belgium), Ernst Eberlein (University of Freiburg, Germany), Monique Jeanblanc (Université d'Evry Val d'Essonne, France), Ragnar Norberg (SAF, Université Lyon 1, France), Michel Vellekoop (University of Amsterdam, the Netherlands), Noel Veraverbeke (Universiteit Hasselt, Belgium) and the chair Griselda Deelstra (Université Libre de Bruxelles, Belgium),* for the excellent scientific support, for their presence at the meeting and for chairing sessions. We also thank *Wouter Dewolf (Ghent University, Belgium),* for the administrative work.

We are very grateful to our sponsors, namely the Royal Flemish Academy of Belgium for Science and Arts, the Research Foundation ─ Flanders (FWO), the Scientific Research Network (WOG) “Stochastic modelling with applications in finance”, le Fonds de la Recherche Scientifique (FNRS), KBC Bank en Verzekeringen, and the BNP Paribas Fortis Chair in Banking at the Vrije Universiteit Brussel and Université Libre de Bruxelles. Without them it would not have been possible to organise this event in this very enjoyable and inspiring environment.

The growing success of the meeting encourages us to continue the organisation of this contactforum to create future opportunities for exchanging ideas and results in this fascinating research field of actuarial and financial mathematics.

The editors:
Griselda Deelstra
Ann De Schepper
Jan Dhaene
Wim Schoutens
Steven Vanduffel
Michèle Vanmaele
David Vyncke

The other members of the organising committee:
Jan Annaert
Pierre Devolder
Pierre Patie
Paul Van Goethem

INVITED TALKS

VARIABLE ANNUITIES AS LIFE INSURANCE PACKAGES: A UNIFYING APPROACH TO THE VALUATION OF GUARANTEES

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Abstract

Variable Annuities (VA) package living and death benefits, eventually aiming at constructing a post-retirement income, while offering a number of possible guarantees in respect of financial and mortality/longevity risks. In many circumstances, the policyholder keeps access to her fund, and can in particular cash money beyond the benefits already set. Apart from financial and mortality/longevity risks, the insurer is therefore exposed also to the risk of unexpected decisions of the policyholder. We suggest to adopt a comprehensive approach to these different risk sources, and we describe a unifying valuation framework of the guarantees under quite general assumptions.

Keywords: Post-retirement income, GMxB, Guaranteed income drawdown, Optimal surrender time, Fair value, Least Square Monte Carlo.

1. INTRODUCTION

Life annuities and pension products usually involve a number of guarantees, such as minimum accumulation rates, minimum annual payments or a minimum total payout. Packaging different types of guarantees is the feature of the so-called variable annuities. Basically, these products are unit-linked investment policies providing a post-retirement income. The guarantees, commonly referred to as GMxBs (namely, Guaranteed Minimum Benefits of type "x"), include minimum benefits both in case of death and survival.

To some extent, the term variable annuity is generic; different benefit arrangements are labeled in this way. Indeed, the variety of what currently is called variable annuity is very rich. This witnesses the advantages of packaging life insurance benefits: under the same policy, the policyholder gets protection against several lifetime risks, and finds long-term investment opportunities as well; on the other hand, the insurer may gain some natural hedging effects among some risks, in particular between mortality and longevity risks.

A description of the main characteristics of variable annuity products, and the market trends, is provided by Ledlie et al. (2008). A number of contributions about possible approaches to the pricing and hedging of the most common guarantees are available in the literature. Recently, special attention has been addressed to withdrawal benefits, thanks to the fact that, when appropriately managed, they can provide a flexible post-retirement income. With regard to the pricing of this class of benefits, Milevsky and Salisbury (2006) assess their cost and compare their findings with prices quoted in the market; Dai et al. (2008) develop a singular stochastic control model, and investigate the optimal withdrawal strategy for a rational policyholder; Chen and Forsyth (2008) describe an impulse stochastic control formulation, while Chen et al. (2008) explore the effect of alternative policyholder behaviours. In all such contributions, most of the attention is addressed to financial risks. The valuation of a variable annuity providing a guaranteed minimum death benefit and a guaranteed minimum withdrawal is the target of Bélanger et al. (2009), while general frameworks for the pricing of guarantees under the assumption of an optimal policyholder behaviour are described by Bauer et al. (2008) and Bacinello et al. (2011). The case of lifelong withdrawals is dealt with by Holz et al. (2007).

After describing the most common designs of GMxB's, we address their fair valuation. Following Bacinello et al. (2011), we adopt a unifying approach to the assessment of different risks originating from the guarantees. In particular, we model jointly financial and mortality/longevity risks. We also address possible withdrawals made by the policyholder, outside of what contractually specified. In this respect, we assume alternatively a passive behavior of the policyholder ("static" approach) and a semi-active behavior ("mixed" approach). None of such assumptions is fully realistic; nevertheless, assuming a completely active behavior of the policyholder ("dynamic" approach) would originate major computational difficulties. Conversely, some bounds for the intrinsic value of the fair fee for the various guarantees are obtained, which may be useful for setting prices in practice.

The paper is arranged as follows. In section 2 we describe the structure of the GMxB's, referring in particular to the most common designs available in the European market. In section 3 we define the valuation framework; in particular, we illustrate the possible approaches to the modeling of the policyholder behavior. In section 4 we provide some numerical investigations of the fair value of the contract. Finally, in section 5 we conclude with some closing remarks.

2. THE STRUCTURE OF GMxB's

A variable annuity is a fund-linked contract, with rider benefits in the form of guarantees on the policy account value; such riders become attainable either upon death or at specified dates. In the latter case, some riders are available just in face of specific policyholder's choices (as we explain below).

Available guarantees are referred to as GMxB, where 'x' stands for the class of benefits involved. According to a first grouping, guarantees are arranged as:

- Guaranteed Minimum Death Benefits (GMDB);
- Guaranteed Minimum Living Benefits (GMLB).

The second class can be further detailed as follows:

- Guaranteed Minimum Accumulation Benefits (GMAB);
- Guaranteed Minimum Income Benefits (GMIB);
- Guaranteed Minimum Withdrawal Benefits (GMWB).

Premiums, arranged on a single recurrent basis, are invested into the reference fund (selected by the policyholder, out of the basket offered by the insurer), and they accumulate, net of fees, in the policy account value. Fees (covering the cost of guarantees, as well as asset management, administrative and other costs) are charged year by year through a reduction of the policy account value. They are typically expressed as a percentage of the policy account value; some upfront costs may also be applied upon premium payment. Usually, the policyholder has the possibility to add or remove some guarantees after policy issue. Accordingly, the corresponding fees start or stop being charged.

We now define in detail the main GMxB's. We refer to a contract issued at time 0, with an accumulation period of T years, $T \geq 0$ (usually T corresponds to the retirement time). The GMDB and GMAB can be cashed before time T (some insurers are willing to provide the GMDB also after time T, but up to some maximum age, say 75 years); the GMIB and GMWB can be cashed after time T, as they provide the post-retirement benefit. If $T = 0$, only post-retirement benefits are involved.

Let A_t denote the policy account value at time t. Clearly, A_t depends on the premiums invested, the benefits paid, the performance of the reference fund, as well as on the cost of the guarantees.

Under a GMDB, the benefit paid upon death (provided that this event occurs prior to the stated maturity) is given by

$$b_t^D = \max\{A_t, G_t^D\}, \tag{1}$$

where G_t^D is the guaranteed amount. Examples of G_t^D are as follows: return of premiums; roll-up of premiums, at a specified guaranteed interest rate; highest policy account value recorded at some prior specified dates (ratchet guarantee); policy account value recorded at the latest of some prior specified dates (reset guarantee). In all examples, the guaranteed amount is suitably modified in case of partial withdrawals. The cost of the guarantee is charged to the policy account value until time T, or as long as the policy stays in-force (because of death or surrender, the contract could terminate prior to time T).

Under a GMAB, at some specified date (typically, time T, as we assume), the insured, if alive, receives the following benefit:

$$b_T^A = \max\{A_T, G_T^A\}, \tag{2}$$

where G_T^A is the guaranteed amount. Examples of G_T^A are similar to those for the GMDB, apart from the reset guarantee (which, within a GMAB, simply allows to postpone the maturity date T).

Similarly to the GMDB, the cost of the guarantee is charged to the policy account value until time T, or (if the contract terminates earlier) as long as the policy stays in-force.

The GMIB provides a lifetime annuity from time T. The guarantee may be arranged in two different ways.

- The guarantee concerns the amount to be annuitised. The post-retirement income is defined as follows:
$$b^I = \eta \max\{A_T, G_T^I\}, \tag{3}$$
where G_T^I is the guaranteed amount to be annuitised, defined similarly to the GMAB. The quantity η is the annuitisation rate, defined at time T according to the prevailing market conditions.

- The guarantee concerns the annuitisation rate. The post-retirement income is defined as follows:
$$b^I = A_T \max\{\eta, g\}, \tag{4}$$
where g is the guaranteed annuitisation rate. This case is usually referred to as the Guaranteed Annuity Option (GAO).

It is implicit in both definitions the assumption that the GMIB will be exercised if it is in the money, neglecting subjective preferences regarding annuitisation vs. bequest, as well as asymmetric information. In both cases, if the annuity is fixed, then b^I is the flat annual income; conversely, if the payments are indexed to the insurer's profit, i.e. the annuity is participating, b^I is just the initial amount of the annual income, which is then subject to revaluation. Of course, arrangements with annuity payments other than annual are possible. We also note that, in principle, it is possible to arrange the GMIB so that both the amount to be annuitised and the annuitisation rate are guaranteed, but this would be very expensive. The cost of the GMIB is charged before annuitisation.

When exercised, the GMIB implies pooling effects; thus, after annuitisation the policyholder looses access to the policy account value. This means, in particular, that upon death of the annuitant no money is available to her estate, unless some death benefits have been explicitly underwritten. The following are examples of arrangements, alternative to a straight lifetime annuity, which provide also a death benefit: annuity-certain with maturity $T' > T$ (in which payments are not contingent on the lifetime; considering the expected lifetime of the policyholder, a long duration $T' - T$ is usually selected for the annuity, say 20-25 years); annuity-certain with maturity $T' > T$ (where $T' - T$ usually ranges from 5 to 10 years), followed by a deferred whole-life annuity if the insured is alive at time T'; life annuity with capital protection (or money-back annuity), in which in case of death prior to time $T' > T$ (with $T' - T$ usually ranging from 5 to 10 years) the insurer pays back the annuitised amount net of the annual payments already made.

It is worthwhile to mention that what is described above summarizes just some of the GMIB arrangements available in the market; indeed, several variants are offered by insurers.

Also the variety of GMWBs available in the market is very rich. Basically, the GMWB is a guaranteed income drawdown. The guarantee concerns the periodic payment and the duration of the income stream. At specified dates (namely, every year, every month, and so on), the policyholder can withdraw the amount
$$b_t^W = \beta_t W_t, \tag{5}$$

where W_t is the so-called base amount (given, for example, by the account value when the guarantee is selected), while β_t is the guaranteed withdrawal rate. At specified dates (e.g., at every policy anniversary), the base amount may step up to the current policy value, if there is a ratchet provision (such a guarantee may be lifetime or limited to a given period, say 10 years); obviously, the base amount is suitably modified if the policyholder withdraws more or less than what is contractually specified. Indeed, the guaranteed payment can alternatively be the exact, the maximum or the minimum amount that the policyholder is allowed to withdraw. The duration of the withdrawals may be fixed (e.g., 20 years) or lifetime. In the former case, if at maturity the account value is positive, it is paid back to the policyholder or, alternatively, the contract stays in-force until the exhaustion of the policy account value. The cost of the guarantee is charged to the policy account value during the withdrawal period. The policyholder keeps access to her fund.

From the descriptions above, it emerges that the GMAB and GMDB are similar to what is offered in endowment insurance contracts, apart from the possible range of guarantees, which is wider within variable annuities. The GMIB is, usually, like a traditional life annuity. The GMWB is the real novelty of variable annuities in respect of traditional life annuities. Even if a GMIB and a GMWB can be arranged so that they become similar (as it is suggested by some of the variants described above), they differ basically in respect of: the duration of the annuity (which is usually lifetime in the GMIB), the accessibility to the policy account value (just for the GMWB) and the features of the reference fund (which can be unit-linked in the GMWB, given that in principle the financial risk is borne by the policyholder, while it is typically fixed-interest, and possibly participating, in the GMIB, given that in this case the financial risk is taken by the insurer). As mentioned above, the presence of death benefits also in the GMIB, a lifetime duration for the withdrawals in the GMWB and other possible features may reduce a lot the differences between the GMIB and the GMWB.

3. THE FAIR VALUE OF THE CONTRACT: THE VALUATION FRAMEWORK

Adopting the well-known approach to the valuation of the market price of a security (see, e.g., Duffie (2001)), we assess the fair value of a variable annuity contract as the expected present value of its cash flows. Discounting is performed at the risk-free rate, while the expectation is taken with respect to a suitably risk-adjusted probability measure. The incompleteness of insurance markets implies that infinitely many such probabilities exist. We assume that the insurer has picked out a specific probability for valuation purposes. From now on, all random variables and processes will be considered under this probability. The expectation conditional on information available at time t is denoted as $E_t[\cdot]$. We let r_t be the risk-free force of interest at time t, and

$$M_t = \mathrm{e}^{\int_0^t r_u \, du}, \quad t \geq 0,$$

the value at time t of 1 euro invested at time 0 in a money market account yielding the risk-free rate r.

For brevity, we refer to a contract issued with a single premium; the more general case of recurrent premiums may be easily obtained. We let P denote the single premium at time 0, net of upfront costs. We further assume that all possible guarantees are selected at time 0 and are maintained as long as the contract stays in-force.

In order to project the future cash flows of a variable annuity contract, we need assumptions about the dynamics of the reference fund, the mortality rates and the policyholder behavior in regard of (partial) withdrawals (outside of what contractually specified). Whatever are such assumptions, the fair value of the contract at time t is defined as follows

$$V_t = E_t\left[\int_t^\infty \frac{M_t}{M_u}\,\mathrm{d}B_u\right], \quad t \geq 0, \tag{6}$$

where B_t represents the total cumulated amount of the benefits paid up to time t, which can be specified in detail once we have modeled the policyholder behavior. Three approaches can be adopted in this regard: a static, a mixed and a dynamic approach.

1. According to the so-called static, or passive, approach (see, e.g., Milevsky and Salisbury (2006)), it is assumed that the policyholder just cashes the benefits contractually specified, with no extra withdrawal. Thus, in particular: the policyholder does not withdraw any fund from her account during the accumulation period or during the payout phase of a GMIB rider; if the contract contains a GMWB rider, the policyholder withdraws exactly the amounts contractually specified; the contract is never surrendered.

In this case, the total cumulated benefit amount B_t is defined as follows:

$$B_t = (B^L_{\tau^-} + b^D_\tau)\,1_{\tau\leq t} + B^L_t\,1_{t<\tau}, \quad t \geq 0, \tag{7}$$

where τ is the residual lifetime of the insured, 1_E is the indicator of the event E (which takes value 1 if E is true, and 0 otherwise), b^D_τ is the death benefit paid upon death and B^L_t ($B^L_{\tau^-}$) is the cumulated living benefit paid up to time t (up to death). Examples of B^L_t are as follows:

- under a GMAB:
  $B^L_t = b^A_T\,1_{t\geq T}$;
- under a GMIB with lifetime payments, and payment dates $(T \leq)\; T_1 < T_2 < \ldots$:
  $B^L_t = b^I\,\sum_{i=1}^\infty 1_{t\geq T_i}$;
- under a GMWB with temporary withdrawals independent of survival, and withdrawal dates $(T \leq)\; T_1 < T_2 < \cdots < T_m\; (\leq T')$:
  $B^L_t = \sum_{i=1}^m b^W_{T_i}\,1_{t\geq T_i} + \max\{A_{T'}, 0\}\,1_{t\geq T'}$.

As described in section 2, the benefits b^A_t, b^I_t, b^W_t and b^D_t depend on the policy account value (see (1)–(5)). So, in order to project the amount of the benefits, we still need to assess the policy account value. Let S_t denote the unit value at time t of the reference fund backing the contract, and φ the proportional fee rate applied to the account value in order to recover the cost of all the guarantees. We have $A_0 = P$, while the instantaneous evolution of the in-force policy account value is as follows:

$$\mathrm{d}A_t = \begin{cases} A_t\,\dfrac{\mathrm{d}S_t}{S_t} - \varphi\,A_t\,\mathrm{d}t - \mathrm{d}B^L_t & \text{if } A_t > 0;\\ 0 & \text{otherwise.}\end{cases} \tag{8}$$

In some situations, the fair value (6) can be expressed in closed-form, once a very natural assumption of stochastic independence between financial and mortality/longevity risk factors is made. This is the case of the celebrated single premium arrangement analysed by Brennan and Schwartz (1976) and Boyle and Schwartz (1977), where the contract contains both a GMDB and a GMAB rider. However, if more sophisticated assumptions do not allow to obtain such closed-form formulae, a straightforward application of the Monte Carlo method can be carried out in order to value the expectation in (6).

Since V_t is net of fees, (6) gives the value of the variable annuity contract as a function of the fee rate φ. Of course the contract is fairly priced if and only if

$$V_0 \equiv V_0(\varphi) = P, \tag{9}$$

so that a fair fee rate φ^* is implicitly defined as a solution of (9).

2. Let us now assume a semiactive behavior of the policyholder (mixed approach): as in the static approach, the policyholder just cashes the benefits contractually stated; however, unlike the static approach she can decide, at any time, to surrender the contract.

We denote by b_t^S the surrender value, given by the policy account value net of some penalty, that the policyholder receives in case of surrender at time $t > 0$. If surrender is not admitted for some t, e.g. during the payout phase of a GMIB rider, then $b_t^S = 0$.

The instantaneous evolution of the in-force policy account value is still defined by (8); however, the surrender value must be added to the total cumulated benefits defined in (7). To this end, let λ denote the surrender time, conventionally equal to τ if the policyholder never surrenders the contract. Given λ, the total cumulated benefits up to time t, denoted by B_t^λ, are now

$$B_t^\lambda = (B_{\tau^-}^L + b_\tau^D)\, 1_{\tau \le \min\{t,\lambda\}} + B_t^L\, 1_{t < \min\{\tau,\lambda\}} + (B_\lambda^L + b_\lambda^S)\, 1_{\lambda < \tau, \lambda \le t}, \quad t \ge 0.$$

The corresponding value of the contract, which is now denoted by V_t^λ, is

$$V_t^\lambda = E_t\left[\int_t^\infty \frac{M_t}{M_u}\, \mathrm{d}B_u^\lambda\right], \quad t \ge 0.$$

Finally, the fair value at time 0 of the contract (net of insurance fees) is given by

$$V_0 = \sup_\lambda V_0^\lambda, \tag{10}$$

where the supremum is taken with respect to all possible surrender times. Once again, the fair fee rate φ^* is implicitly defined by (9). We note that in (10) we assume implicitly in respect of the surrender decision an optimal behavior of the policyholder (corresponding to the worst case for the insurer); individual preferences are disregarded, while they could lead to suboptimal choices, and then a lower value of the contract.

The supremum in (10) needs to be evaluated by means of a numerical approach. Among the possible methods that have been proposed in the literature to solve such kind of problems (binomial or multinomial trees, partial differential equations with free boundaries, Least Squares Monte Carlo), we choose the Least Squares Monte Carlo method. Our problem fits perfectly in the general framework dealt with by Bacinello et al. (2010), so that we refer to their paper for an accurate description of the philosophy underlying this method and the valuation algorithm. For a brief description of the Least Square Monte Carlo algorithm adapted to our framework, we refer to Bacinello et al. (2011).

3. We now provide some remarks on the dynamic approach to policyholder behavior. In this case, it is assumed that the policyholder is active: she withdraws amounts not necessarily coinciding with those contractually specified under a GMWB; in particular, she can decide not to withdraw, or to surrender the contract; partial withdrawals or surrender decisions could be made on dates not coinciding with those contractually specified, as well as during the accumulation period. In all of these cases, some penalties are applied by the insurer, while guaranteed amounts (if present) are accordingly reduced.

The fair value of the contract must be assessed in respect of all possible (and admitted) withdrawal strategies. A withdrawal strategy is modeled as a stochastic process specifying the amount to withdraw at each date (provided the insured is alive), including possibly the surrender value. A strategy is admitted if it complies with contractual constraints. Let θ be the selected withdrawal strategy; we denote by B_t^θ the total cumulated benefits up to time t; since we are not providing numerical assessments of the fair value of the contract under this assumption, we do not define B_t^θ in detail (we refer, instead, to Bacinello et al. (2011)). Given θ, the contract value at time t is defined as follows:

$$V_t^\theta = E_t\left[\int_t^\infty \frac{M_t}{M_u}\,\mathrm{d}B_u^\theta\right], \quad t \geq 0,$$

while the fair value of the contract at time 0 is

$$V_0 = \sup_\theta V_0^\theta, \tag{11}$$

where the supremum is taken with respect to all admitted withdrawal strategies. We note that (11), similarly to the mixed approach, gives the value of the contract under the assumption of an optimal policyholder behavior; depending on individual preferences, the real value of the insurer's liability could be lower. As already mentioned, in this paper we do not offer specific results following this fully dynamic approach, which of course requires a numerical scheme. Some results concerning a contract with a GMWB rider are presented, e.g., by Milevsky and Salisbury (2006), Dai et al. (2008), Chen et al. (2008) and Chen and Forsyth (2008).

In order to compare the values obtained under the three approaches to the policyholder behavior, we denote by V_0^{static}, V_0^{dynamic} and V_0^{mixed} the contract values at time 0 defined respectively by (6), (10) and (11). By their very definition, it turns out

$$V_0^{\text{static}} \leq V_0^{\text{mixed}} \leq V_0^{\text{dynamic}}.$$

Indeed, from the insurer's point of view, the dynamic approach assumes the worst case scenario since the policyholder can choose among all withdrawal strategies and, in particular, the surrender time. In the mixed approach, instead, the policyholder can choose only the surrender time, so that her optimal strategy is selected within a subset of what considered by a dynamic agent. Finally, the static approach defines a single, specific, withdrawal strategy included in the previous subset. As a consequence, the proportional fees that have to be applied to the account value to make the contract fair are ordered in the same way: they are the highest with the dynamic approach and the smallest with the static one.

4. THE FAIR VALUE OF THE CONTRACT: SOME EXAMPLES

We provide some numerical investigation on the fair value of (single premium) variable annuity arrangements. Due to the importance of individual post-retirement benefits (which, in many countries, are needed to complement the reduced social security benefit and the benefit provided by defined contribution pension plans), we focus in particular on immediate post-retirement guarantees. We address the following arrangements, which differ one from the other for the duration of the annuity and the possible presence of death benefits:

case a: GMWB with maturity 20 years;

case b: lifetime GMWB;

case c: lifetime GMWB, joint to a GMDB with maturity 10 years and guaranteed amount given by the single premium net of the GMWB benefits totally paid up to death.

In all cases, the entry age is $x = 65$, while (according to the previous notation) $T = 0$; hence, just the post-retirement period is focused. The base amount used to define the annual income b_t^W (see (5)) is $W_t = A_0 = P$; the guaranteed withdrawal rate is: $\beta_t = \frac{1}{20} = 5\%$ in case a; $\beta_t = 4.5\%$ in case b (such a rate corresponds, approximately, to the reciprocal of the expected residual lifetime obtained under the assumed mortality model, which is defined below); $\beta_t = 4\%$ in case c (the lower guaranteed withdrawal rate in this case, when compared to case b, is justified by the presence of the GMDB). In case of surrender at time t (event that we address within the mixed approach), a surrender penalty rate p is applied. The policyholder cashes the benefit $b_t^S = (1-p)\,A_t$ if t is not a withdrawal date, and $b_t^S = b_t^W + (1-p)\,\max\{A_t - b_t^W, 0\}$ otherwise.

We now briefly describe the financial and mortality model adopted in the valuation. We refer to the framework presented in Bacinello et al. (2010) and Bacinello et al. (2011). All processes are specified under the selected risk-adjusted probability measure.

We assume that the risk-free force of interest follows the square root process:

$$\mathrm{d}r_t = \xi_r\,(\zeta_r - r_t)\,\mathrm{d}t + \sigma_r\,\sqrt{r_t}\,\mathrm{d}Z_t^r.$$

We then assume that the unit value of the reference fund satisfies a diffusion with stochastic volatility, namely the dynamics

$$\mathrm{d}\log S_t = \left(r_t - \frac{1}{2}\,K_t\right)\mathrm{d}t + \sqrt{K_t}\,\left(\rho_{SK}\,\mathrm{d}Z_t^K + \sqrt{1-\rho_{SK}^2}\,\mathrm{d}Z_t^S\right),$$

where the variance K_t follows the square root process

$$\mathrm{d}K_t = \xi_K\,(\zeta_K - K_t)\,\mathrm{d}t + \sigma_K\,\sqrt{K_t}\,\mathrm{d}Z_t^K.$$

We assume a stochastic force of mortality, following the mean reverting square root process

$$\mathrm{d}\mu_t = \xi_\mu\,(\hat{\mu}(t) - \mu_t)\,\mathrm{d}t + \sigma_\mu\,\sqrt{\mu_t}\,\mathrm{d}Z_t^\mu, \tag{12}$$

where $\hat{\mu}(t) = c_1^{-c_2}\,c_2\,(x+t)^{c_2-1}$ is the deterministic Weibull intensity. In the following, we denote by $\hat{\tau}$ the residual lifetime having force of mortality $\hat{\mu}$. The standard Brownian motions

Z^r, Z^K, Z^S, Z^μ are assumed to be independent. The residual lifetime τ is then linked to μ through

$$\tau = \inf\left\{t \geq 0 \; : \; \int_0^t \mu_s \, \mathrm{d}s > \mathcal{E}\right\},$$

where $\mathcal{E}$ is a unit exponential random variable independent of $(\mu_t)_{t\geq 0}$. The force of mortality μ_t drives the instantaneous probability of death at time t conditional on survival for an individual aged x at time 0. The unconditional survival probability is given by

$$P(\tau > t) = E\left[\mathrm{e}^{-\int_0^t \mu_s \, \mathrm{d}s}\right]$$

and can be computed in closed form given the affine nature of μ (see Biffis and Millossovich (2006)). We note that, according to (12), the force of mortality μ is pushed toward the moving target $\hat{\mu}$ and the dynamics is noised by the Brownian term, amplified by the size of μ.

The parameters underlying these processes are listed in Table 1. We do not deal with the calibration of the model to market data, as we just aim at illustrating how to perform a fair valuation of the contract allowing for several risk sources. The coefficients c_1 and c_2 defining the Weibull intensity $\widehat{\mu}$ were obtained by fitting the survival probabilities for a male aged 65 implied by the projected life table IPS55, commonly used in the Italian annuity market.

r	K	S	A	μ
$r_0 = 0.03$	$K_0 = 0.04$	$S_0 = 100$	$A_0 = 100$	$\mu_0 = \widehat{\mu}(0)$
$\xi_r = 0.60$	$\xi_K = 1.50$	$\rho_{SK} = -0.70$		$\xi_\mu = 0.50$
$\zeta_r = 0.03$	$\zeta_K = 0.04$			$\sigma_\mu = 0.03$
$\sigma_r = 0.03$	$\sigma_K = 0.40$			$c_1 = 90.50$
				$c_2 = 10.49$

Table 1: Parameters used in the simulation.

Results relating to the static approach were obtained by averaging over 10 sets of $10\,000$ simulations each, while those relating to the mixed approach by averaging over 10 sets of $20\,000$ simulations each. In the Least Square Monte Carlo algorithm we have employed 4 state variables (K, A, r and μ) and up to 4-th degree powers as basis functions (see Bacinello et al. (2011) for more details).

In Table 2 we quote the fair value of a GMWB contract with duration 20 years (i.e., case a) obtained under the static (last column of the table) and the mixed approach (central columns). Alternative values have been set for the fee rate φ, and the surrender penalty rate p. As expected, the cost of the liability obtained under the mixed approach is higher, as an optimal decision is taken by the policyholder in respect of the surrender time. The single premium is set to $P = 100$. When the contract value is higher than 100, it means that either the fee rate or the penalty rate are too low; vice versa, when the contract value is lower than 100. For example, while the fee rate φ turns always to be too high under the static approach (so that we can conclude for the fair fee rate $\varphi^* < 1\%$), under the mixed approach and a surrender penalty rate $p = 1\%$ we can conclude for the fair fee rate $2\% < \varphi^* < 3\%$. Similar comments can be made in respect of Tables 3 and 4, where the fair value of the contract for cases b and c is quoted. We point out that the results in the three tables are not directly comparable, due to the different guaranteed withdrawal rates. Clearly, in setting such a rate the insurer has to look for the trade-off between the guaranteed withdrawal rate and the fair fee rate which better satisfies commercial aspects.

	V_0^{mixed}						V_0^{static}
φ	$p = 0\%$	$p = 1\%$	$p = 2\%$	$p = 3\%$	$p = 4\%$	$p = 5\%$	
1%	103.921	103.261	102.636	102.021	101.446	100.923	98.696
2%	101.260	100.624	99.912	99.270	98.588	97.908	91.536
3%	99.823	99.042	98.315	97.543	96.802	96.077	86.238
4%	98.555	97.793	97.040	96.275	95.551	94.818	82.427
5%	97.537	96.752	95.963	95.187	94.446	93.706	79.755
6%	96.543	95.749	94.977	94.213	93.465	92.729	77.936

Table 2: Fair value of a GMWB with duration 20 years (case a).

	V_0^{mixed}						V_0^{static}
φ	$p = 0\%$	$p = 1\%$	$p = 2\%$	$p = 3\%$	$p = 4\%$	$p = 5\%$	
1%	109.798	109.171	108.525	107.953	107.391	106.825	106.825
2%	104.296	103.733	103.138	102.527	101.934	101.307	99.759
3%	102.397	101.729	101.129	100.535	99.946	99.379	94.810
4%	100.807	100.170	99.528	98.955	98.313	97.714	91.350
5%	99.440	98.804	98.165	97.562	96.960	96.353	88.918
6%	98.245	97.604	96.973	96.382	95.775	95.170	87.195

Table 3: Fair value of a GMWB with lifetime duration (case b).

	V_0^{mixed}						V_0^{static}
φ	$p = 0\%$	$p = 1\%$	$p = 2\%$	$p = 3\%$	$p = 4\%$	$p = 5\%$	
1%	105.358	104.655	103.992	103.371	102.717	102.115	102.115
2%	101.685	100.971	100.273	99.569	98.846	98.198	94.095
3%	99.873	99.159	98.434	97.745	97.066	96.343	88.390
4%	98.320	97.626	96.911	96.214	95.500	94.827	84.344
5%	97.015	96.288	95.552	94.849	94.152	93.470	81.471
6%	95.820	95.099	94.386	93.677	92.977	92.267	79.416

Table 4: Fair value of a GMWB with lifetime duration joint to a GMDB with maturity 10 years (case c).

5. FINAL REMARKS

Variable annuities have several appealing features, as they merge the most attractive characteristics of unit-linked and participating life insurance contracts: dynamic investment opportunities, protection against financial risks, benefits in case of early death or high longevity. Further, they offer the opportunity to arrange a satisfactory trade-off between annuitisation needs and bequest preferences. Pricing tools used in practice seem to be not appropriate, as they mainly focus just on a part of the risks (typically, the financial risks), assuming overall simplified models. On the contrary, the variety of the risk sources requires to adopt a unifying framework, as we suggest in this paper. There are still many open issues to be discussed. As to the policy design, it is interesting to consider guarantees relating to the health status, with particular regard to long-term benefits in the post-retirement period. From a modeling point of view, further work is required in respect of the policyholder behavior, allowing for individual preferences. In a risk management perspective, it is interesting to examine the risk profile of the insurer as a function of the alternative ways of arranging the benefits; in particular, natural hedging effects, or a worsening of the overall risk position, originated by some benefit combinations should be investigated.

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WHY RUIN THEORY SHOULD BE OF INTEREST FOR INSURANCE PRACTITIONERS AND RISK MANAGERS NOWADAYS

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Ruin theory concerns the study of stochastic processes that represent the time evolution of the surplus of a stylized non-life insurance company. The initial goal of early researchers of the field, Lundberg (1903) and Cramér (1930), was to determine the probability for the surplus to become negative. In those pioneer works, the authors show that the ruin probability $\psi(u)$ decreases exponentially fast to zero with initial reserve $u \geq 0$ in numerous cases when the net profit condition is satisfied: if the insurance company receives premium continuously at a deterministic rate $c >$ and pays for claims that are described by a compound Poisson process, for all $u \geq 0$ we have an upper bound for the ruin probability $\psi(u) \leq e^{-Ru}$, as well as information on the asymptotic behaviour, because $\psi(u) \sim Ce^{-Ru}$ as $u \to +\infty$, where $0 < C \leq 1$. This result is valid for light-tailed claim amounts, i.e. when the probability of very large claims decreases fast enough. This condition is satisfied in the particular case where claims amounts are bounded, which is often true in practice. Following the approach of Gerber (1974), it is possible to link the Cramr-Lundberg adjustment coefficient R with the risk aversion coefficient a. If one measures a random claim amount X thanks to indifference pricing method (which means that the insurer does not show any preference between not insuring the risk and bearing the risk after receiving premium π), with exponential utility function $u(x) = (1 - e^{-ax})/a$, the insurer would ask for premium[1]

$$\pi = \frac{1}{a} \ln \left(E \left(e^{aX} \right) \right).$$

Gerber (1974) notes that if the insurer determines the premium following this principle, then the Cramér-Lundberg adjustement coefficient R is identical to the risk aversion parameter a. Conversely, if the insurer wants the ruin probability to decrease exponentially fast, she can use

[1] Denote by $E(Y)$ the mathematical expectation of an integrable random variable X, by $Var(X)$ its variance if X is square integrable. Denote $VaR_{\beta}(X)$ the Value-at-Risk (quantile) of level $\beta \in [0, 1]$ of a general random variable X.

indifference pricing principle with exponential utility function. Note that in this dynamic vision, at first order, the insurer uses a pricing principle that looks like the variance principle $\pi \simeq E(X) + \frac{a}{2}Var(X)$. This is different from the static framework, which consists (like in Solvency II) in studying the probability that the net asset value of the company is negative in one year. If one computes the risk margin thanks to the cost of capital approach, this leads to a theoretical pricing as $E(X) + b\,(VaR_{99,5\%}(X) - E(X))$. This corresponds at first order to the standard error coefficient pricing principle $\pi \simeq E(X) + bq\sqrt{Var(X)}$, where b is a parameter that quantifies cost of capital, and q is a factor that links the standard error coefficient and the 99.5%-Value-at-Risk of X (approximately 3 for a Gaussian distribution, 4 or 5 for heavier tails). Ruin theory thus provides more sustainable valuation principle than the Value-at-Risk approach, because it takes into account liquidity constraints and penalizes large position sizes.

In risk management, insurance companies start to set risk limits: more precisely, they want to guarantee that the Solvency Capital Requirement (SCR) coverage ratio stays above a certain level with a large enough probability. Modeling the evolution of the SCR coverage ratio is of course delicate. Internal models (that study the one year change in net asset value) are already very complex and require large computation times. On the average term, insurers often merely study solvency in some adverse scenarios, without trying to affect probability to each of those scenarios. Ruin theory does not offer a precise, miraculous answer to this question, but it may provide interesting insight thanks to different situations for which the ruin probability is known explicitly or can be approximated. Note that the zero surplus level corresponds then to the minimum SCR coverage ratio level in that case. Finite-time ruin probabilities have been studied among others by Picard and Lefèvre (1997), and Ignatov et al. (2001). The probability of ruin at inventory dates has been studied by Rullière and Loisel (2003). Researchers in ruin theory currently work on models with credibility adjusted premium, with tax payments, with correlations and correlation crises between claim amounts, as well as the ability for the insurer to invest into risky assets or to transfer part of its risks. Less binary risk and profit indicators are also considered. For regularly varying claim size distributions (Pareto distribution for example), Embrechts and Veraverbeke (1982) have shown that the ruin probability decreases more slowly with u:

$$\psi(u) \sim K u^{-\alpha+1},$$

where $\alpha > 1$. In several models with a non diversifiable and no compensable risk driver, Albrecher et al. (2011) and Dutang et al. (2012) show that the ruin probability admits a positive limit as $u \to +\infty$:

$$\psi(u) - A \sim \frac{B}{u},$$

where $0 < A < 1$ and $B > 0$ are constant numbers. Here, ruin should be understood in a broader sense, economic ruin or switch to run-off mode before being completely ruined. This corresponds to the idea that capital is not always the answer and that the capacity to react fast is a key element of efficient risk management. The book by Asmussen and Albrecher (2010) contains most references of papers dealing with ruin theory.

Another classical problem of ruin theory is to determine optimal dividend strategies. In Switzerland, if ruin were not a problem, it would not be efficient to pay dividends, because they are taxed.

It would be better to let the stock price increase faster in the absence of dividends, because capital gains are not taxed. But as ruin may occur, the investor faces the problem of dividend optimization. De Finetti, who was also actuary at Generali, shew in a simplified model that the optimal dividend strategy consisted in paying dividends above some horizontal barrier (which of course increases ruin probability) and computed the optimal barrier level. Dubourdieu (1952) formalized several results on this issue and gave credit to De Finetti for the main ideas (see posterior paper by De Finetti (1957)). In a more general setting, optimal strategies might involve several bands instead of one single barrier. Since the works by Borch (1974) and by Gerber (see for example Gerber (1979)), this subject had been almost forgotten, but has been addressed by numerous papers in the recent years (see the survey by Avanzi (2009) on those issues and by Albrecher et Thonhauser (2009) on optimal control strategies).

This theory could be useful to address the problem of determining appropriate Solvency Capital Requirement coverage ratio target levels. In the new regulation framework Solvency II, in addition to the technical provisions (composed of best estimate of liabilities and of a risk margin), the insurer must have at least the so-called Solvency Capital Requirement (SCR). Most insurers have now to choose a target SCR level, usually comprised between 110% and 200%. Besides, they usually adopt a kind of dividend strategy that corresponds to a refraction strategy: if the SCR coverage ratio becomes higher than a threshold, then the insurer starts to pay part of the excess as dividends. If the SCR coverage ratio overshoots a certain level (250%, say), then all the excess is paid as dividends, which corresponds to reflection from a barrier. For Enterprise Risk Management purposes, it might be interesting to study the probability to become insolvent before 5 or 10 years in a steady regime to check whether the activity would be sustainable in a steady regime, in the absence of change of risk environment. With a first-order approximation, this corresponds to a finite-time ruin problem with a certain dividend strategy, where the ruin level is the 100% coverage ratio level (it is different from the economic ruin level where the net asset value of the company becomes negative). The dynamic balance sheet is illustrated in Figure 1 and the simplified ruin problem is illustrated in Figure 2.

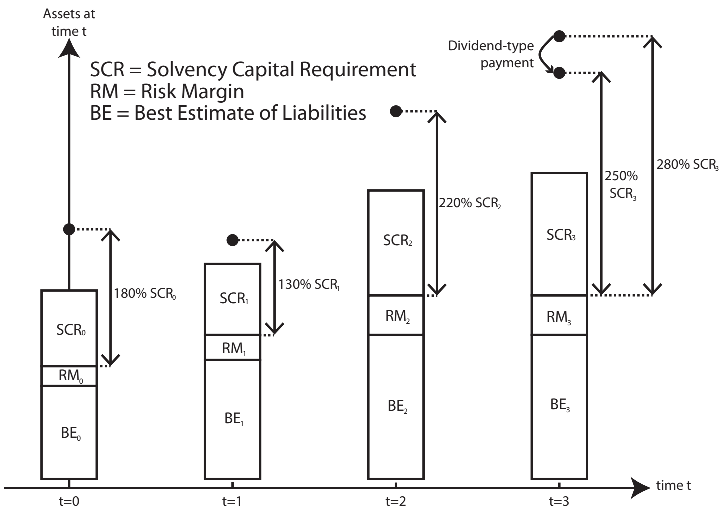

Figure 1: Evolution of economic balance sheet.

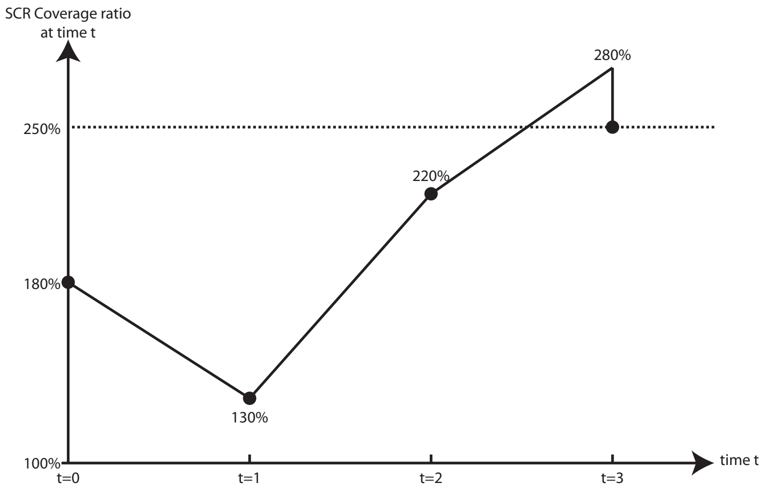

Figure 2: Evolution of SCR coverage ratio.

Nevertheless the simplified view is far from being perfect, because the insolvency threshold depends on the evolution of the assets and liabilities. Of course, the evolution of the economic balance sheet of a company is much more complicated than classical risk models. However, as multi-period risk models are often intractable on a 5-year time horizon in practice, it may be interesting to have benchmarks that come from ruin theory in mind while thinking about the risk appetite implementation.

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RISK BOUNDS, WORST CASE DEPENDENCE, AND OPTIMAL CLAIMS AND CONTRACTS

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Abstract

Some classical results on risk bounds as the Fréchet bounds, the Hoeffding–Fréchet bounds and the extremal risk property of the comonotonicity dependence structure are used to describe worst case dependence structures for portfolios of real risks. An extension of the worst case dependence structure to portfolios of risk vectors is given. The bounds are used to (re-)derive and extend some results on optimal contingent claims and on optimal (re-)insurance contracts.

1. RISK BOUNDS AND COMONOTONICITY

For a risk vector $X = (X_1, \dots, X_n)$ of risks X_i with distributions P_i resp. distribution functions F_i, it is a classical problem to determine (sharp) bounds for a risk functional of the form $E\Psi(X)$ induced by dependence between the components X_i of X. The class of all possible dependence structures is given by the Fréchet class $\mathcal{M}(P_1, \dots, P_n)$ of joint distributions with marginals P_i. For the case of real risks one can consider equivalently the class $\mathcal{F}(F_1, \dots, F_n)$ of joint distribution functions with marginal distribution functions $F_i \sim P_i$.

The sharp upper and lower dependence bounds for the risk function Ψ are given by

$$\begin{aligned} M(\Psi) &= \sup\Big\{ \int \Psi dP; P \in M(P_1, \dots, P_n)\Big\} \\ \text{and} \quad m(\Psi) &= \inf\Big\{ \int \Psi dP; P \in M(P_1, \dots, P_n)\Big\}. \end{aligned} \tag{1}$$

They are called *(generalized) upper resp. lower Fréchet bounds*. Typical risk functionals of interest are, in the case of real risks X_i, risk functionals of the joint portfolio like $(\sum_{i=1}^n X_i - K)_+$, $1_{[t,\infty)}(\sum_{i=1}^n X_i)$ or $\max_i X_i$ leading to bounds for the excess of loss, for the value at risk and for the maximal risk of the joint portfolio.

The following three examples of sharp risk bounds are classical examples of generalized Fréchet bounds.

B1) Sharpness of Fréchet bounds
If X_i are real random variables with distribution functions F_i, $1 \leq i \leq n$, then for the distribution function $F = F_X$ of $X = (X_1, \dots, X_n)$ the following bounds are sharp:

$$F_c(x) := \left(\sum_{i=1}^n F_i(x_i) - (n-1) \right)_+ \leq F(x) \leq F^c(x) := \min_i F_i(x_i). \tag{2}$$

The upper bound $F^c(x)$ is the distribution function of the comonotonic vector $X^c := (F_1^{-1}(U), \dots, F_n^{-1}(U)) = (X_1^c, \dots, X_n^c)$ where $U \sim U(0,1)$. In consequence, the upper bound in (2) is sharp. The lower bound $F_c(x)$ is a distribution function if $n = 2$ and then corresponds to the antithetic (countermonotonic) vector

$$X^{cm} := (F_1^{-1}(U), F_2^{-1}(1-U)) = (X_1^{cm}, X_2^{cm}).$$

The bounds in (2) go back to Fréchet (1951) and Hoeffding (1940) for $n = 2$. The upper and lower bounds were described in Dall'Aglio (1972). Sharpness of the lower bound in (2) was first given in Rü[1] (1981).

B2) Hoeffding–Fréchet bounds
For real random variables X_1 and X_2, Hoeffding (1940) found the following representation of the covariance:

$$\operatorname{Cov}(X_1, X_2) = \iint (F(x,y) - F_1(x)F_2(y))dxdy. \tag{3}$$

Together with the Fréchet bounds in (2) this representation implies the sharp upper and lower Hoeffding–Fréchet bounds:

$$\operatorname{Cov}\left(F_1^{-1}(U), F_2^{-1}(1-U)\right) \leq \operatorname{Cov}(X_1, X_2) \leq \operatorname{Cov}\left(F_1^{-1}(U), F_2^{-1}(U)\right), \tag{4}$$

or, equivalently,

$$EF_1^{-1}(U)F_2^{-1}(1-U)) \leq EX_1X_2 \leq EF_1^{-1}(U), F_2^{-1}(U)). \tag{5}$$

The comonotonic resp. countermonotonic vectors are the unique (in distribution) vectors which attain the upper resp. lower risk bounds in (4) and (5).

If V is a random variable uniformly distributed on $(0,1)$ and independent of X_1 and X_2, we define the distributional transform

$$U_i := F_i(X_i, V) = \tau_{X_i}, \quad i = 1, 2, \tag{6}$$

where $F_i(x, \lambda) := P(X_i < x) + \lambda P(X_i = x)$ are the modified distribution functions. Then we have that

$$U_i \sim U(0,1) \quad \text{and} \quad X_i = F_i^{-1}(U_i) \text{ a.s.} \tag{7}$$

[1]Rüschendorf is abbreviated with Rü in this paper.

In fact the pair (U_1, U_2) is a copula vector of X (see Rü (1981, 2009)). Further the pairs

$$\begin{aligned} &(X_1, F_2^{-1}(F_1(X_1, V))) = (X_1, F_2^{-1}(\tau_{X_1})) \\ \text{and } &(X_1, F_2^{-1}(1 - F_1(X_1, V))) = (X_1, F_2^{-1}(1 - \tau_{X_1})) \end{aligned} \tag{8}$$

are comonotonic resp. countermonotonic pairs with marginal distribution functions F_1 and F_2 and thus attain the upper resp. lower Hoeffding–Fréchet bounds in (5). The interesting point in (8) is that the solution can be written as pair $(X_1, F_2^{-1}(\tau_{X_1}))$ resp. $(X_1, F_2^{-1}(1-\tau_{X_1}))$ with distributional transform $\tau_{X_1} = F_1(X_1, V)$ which is increasing in (X_1, V).

B3) Comonotonic vector as worst case dependence structure
The third classical result concerns sharp upper bounds on the excess of loss. It states that the comonotonic vector $X^c = (F_1^{-1}(U), \dots, F_n^{-1}(U))$ is the *worst case dependence structure* w.r.t. excess of loss. Formulated in terms of convex ordering $\leq_{\text{cx}}$ it says:

$$\text{If} \quad X_i \sim F_i, 1 \leq i \leq n, \text{ then} \quad \sum_{i=1}^n X_i \leq_{\text{cx}} \sum_{i=1}^n F_i^{-1}(U). \tag{9}$$

This result was first established in Meilijson and Nadas (1979) together with the following equivalent representation: For all $d^* \in \mathbb{R}^1$ holds

$$\begin{aligned} \sup_{X_i \sim F_i} E\left(\sum_{i=1}^n X_i - d^*\right)_+ &= E\left(\sum_{i=1}^n F_i^{-1}(U) - d^*\right)_+ \\ &= \Psi_+(d) := \inf_{\sum_{i=1}^n d_i = d^*} \sum_{i=1}^n E(X_i - d_i)_+. \end{aligned} \tag{10}$$

For continuous strictly increasing distribution functions one can choose a solution (d_i^*) of (10) as

$$d_i^* = F_i^{-1}\Big(F_{\sum_{i=1}^n X_i^c}(d^*)\Big). \tag{11}$$

In general, if d^* is a u_0-quantile of $\mathcal{L}(\sum_{i=1}^n X_i^c)$, then d_i^* can be chosen as u_0-quantile of F_i.

As a consequence of (9), one obtains that

$$\Psi\left(\sum_{i=1}^n X_i\right) \leq \Psi\left(\sum_{i=1}^n X_i^c\right) \tag{12}$$

for all law invariant convex risk measures Ψ (see Föllmer and Schied (2004), Burgert and Rü (2006)). Thus the comonotonic risk vector is in this sense a *universal* worst case dependence structure for the joint portfolio.

2. WORST CASE DEPENDENCE FOR RISK VECTORS

In the case that the components X_i of the risk vector X are d-dimensional, $1 \leq d$, there does not exist a universal worst case dependence structure corresponding to the comonotonic vector in

$d = 1$. Several aspects of this problem have been described in Rü (2004) and Puccetti and Scarsini (2010). To each law invariant risk measure Ψ, there corresponds one worst case dependence structure which is described in Rü (2006, 2012).

Let for a density vector $Y = (Y_1, \dots, Y_d)$ with $Y_i \geq 0$, $EY_i = 1$, $1 \leq i \leq d$, with distribution μ

$$\Psi_\mu(X) := \sup\{E\tilde{X} \cdot Y; \tilde{X} \stackrel{d}{=} X\} \tag{13}$$

denote the *max-correlation risk measure* in direction Y (resp. μ) as introduced in Rü (2006). Then $\Psi_\mu(X)$ defines a law invariant convex risk measure defined for risk vectors $X \in \mathbb{R}^d$. Any lsc convex law invariant risk measure Ψ on $L_d^p(P)$, the class of risk vectors with components $X_i \in L^p(P)$, has a representation as

$$\Psi(X) = \sup_{\mu \in A}(\Psi_\mu(X) - \alpha(\mu)), \tag{14}$$

where A is a weakly closed class of scenario measures and $\alpha(\mu)$ is a law invariant penalty function. Thus the max-correlation risk measures play in the multivariate case a similar role as the spectral risk measures in $d = 1$ and are the building blocks of the class of convex, law invariant risk measures.

The worst case dependence structure of a joint portfolio $\sum_{i=1}^n X_i$ with $X_i \in \mathbb{R}^d$, $X_i \sim F_i$ w.r.t. a law invariant convex risk measure Ψ as in (14) is defined as $X_i^* \sim F_i$, $1 \leq i \leq n$, such that

$$\Psi\left(\sum_{i=1}^n X_i^*\right) = \sup_{X_i \sim F_i} \Psi\left(\sum_{i=1}^n X_i\right). \tag{15}$$

Its determination involves two steps:

Step 1) Determine a worst case scenario measure $\mu^* \in A$ solving an optimization problem of the form

$$F_a(\mu^*) = \sup_{\mu \in A} F_a(\mu), \tag{16}$$

where $F_a(\mu) = \sum_{i=1}^n \Psi(X_i) - \alpha(\mu)$ is the sum of the marginal risks. $F_a(\mu)$ depends only on the marginals F_i.

Step 2) Let $X_i^* \sim F_i$, $1 \leq i \leq n$ be μ^*-comonotone, i.e. for some $Y^* \sim \mu^*$

$$X_i^* \sim_{\text{oc}} Y^*, \quad 1 \leq i \leq n. \tag{17}$$

All X_i^* are optimally coupled to the same vector Y^*, $1 \leq i \leq n$, in the L^2-sense, i.e. they solve the classical mass transportation problem

$$E\|X_i^* - Y^*\|^2 = \inf\{E\|X_i - Y\|^2; X_i \sim F_i, Y \sim \mu\}. \tag{18}$$

Step 1) and Step 2) imply that

$$(X_1^*, \dots, X_n^*) \text{ is a worst case dependence structure w.r.t. } \Psi. \tag{19}$$

One could call the vector $X^* = (X_1^*, \dots, X_n^*)$ in analogy to the case $d = 1$ a Ψ*-comonotonic vector*. Some examples like elliptical distributions, Archimedian copulas and location-scale families are discussed in Rü (2006, 2012). In general both steps needed to determine worst case dependent vectors can be done only numerically.

3. APPLICATIONS OF DEPENDENCE BOUNDS

Our aim in this section is to use the classical dependence bounds for risk functionals to derive in a simple and unified way some results on the optimization of financial products and of (re-)insurance contracts.

3.1. Optimal contingent claims

As a step to derive optimal portfolio results as in the classical paper of Merton (1971), He and Pearson (1991a,b) formulated the static problem of optimal claims. This also fitted with economic theory on optimal investments following the Markowitz theory. As reference we mention Merton (1971) and for more recent formulation Dybvig (1988), Dana (2005), Schied (2004), and Föllmer and Schied (2004). The problem of cost efficient options was formulated in Dybvig (1988) and discussed in detail in Bernard and Boyle (2010) and Bernard et al. (2011a,b).

3.1.1. Optimal investment problem

Given an investment (claim) X and a price measure $Q = \varphi \cdot P$ with price density φ w.r.t. P the *optimal investment problem* is formulated as follows:

Find an optimal investment C^* such that

$$E_Q C^* = \int \varphi C^* dP = \inf_{C \leq_{\mathrm{cx}} X} . \tag{20}$$

C^* has the lowest price under all investments C, which are less risky than X in the sense of convex order $\leq_{\mathrm{cx}}$. The minimal price

$$e(X, \varphi) := E_Q C^* \tag{21}$$

is called the *reservation price* in Jouini and Kallal (2001). The following result is stated in Dybvig (1988), Dana (2005), and Föllmer and Schied (2004) in various generality.

Theorem 3.1 (Optimal investment) *Let X be an investment with $F_X = F$ and let φ be a price density. Then the reservation price is given by*

$$e(X, \varphi) = \int_0^1 F_\varphi^{-1}(1-t) F^{-1}(t) dt. \tag{22}$$

An optimal investment is given by

$$C^* = F^{-1}(1 - \tau_\varphi(\varphi; V)), \tag{23}$$

where τ_φ is the distributional transform of φ (see (8)*).*

Proof. By the Hoeffding–Fréchet bounds in (5) for any investment C, we have

$$A_\varphi(C) := \inf_{\tilde{C} \sim C} \int \varphi \tilde{C} dP = \int_0^1 F_\varphi^{-1}(1-t) F_C^{-1}(t) dt. \tag{24}$$

Also by a well-known stochastic ordering result

$$C_1 \le_{\mathrm{cx}} C_2 \text{ implies } \int_0^1 h(t) F_{C_1}^{-1}(t)dt \ge \int_0^1 h(t) F_{C_2}^{-1}(t)dt$$

for decreasing functions h. This implies that

$$\inf_{C \le_{\mathrm{cx}} X} A_\varphi(C) = A_\varphi(X) = e(X,\varphi) = \int_0^1 F_\varphi^{-1}(1-t)F^{-1}(t)dt.$$

The representation of the optimal claim in (23) by the distributional transform follows from the fact that the pair (φ, C^*) attains the lower Fréchet bound (see (8)). ∎

Remark 3.1 *a)* (C^*,φ) *is a pair of antithetic variables. The distribution of the optimal pair is unique and is given by the anticomonotone distribution. Defining*

$$\tilde{C} := E(C^* \mid \varphi) = \int_0^1 F^{-1}(1-\tau_\varphi(\varphi,v))dv, \tag{25}$$

then $\tilde{C} = g(\varphi)$*, where* $g \downarrow$ *is a decreasing function of the price density* φ *alone. Further,* $\tilde{C} \le_{\mathrm{cx}} C$ *and* $E_Q\tilde{C} = E_Q C^*$*. Thus there exists an optimal investment* $C^* = g^*(\varphi)$*, where* $g^* \downarrow$ *is a decreasing function of the price density* φ*.*

b) ***Transformed measure.*** *Defining the transformed measure*

$$Q^* := \varphi^* P \text{ with } \varphi^* := F^{-1}(1-\tau_F(X,V)), \tag{26}$$

then φ^* *is decreasing in* X *and*

$$e(X,\varphi) = E_Q C^* = E_{Q^*} X. \tag{27}$$

Thus the reservation price is identical to the expectation of X *w.r.t. the transformed price measure* Q^**. Then* Q^* *describes a worst case price density for the claim* X*.*

c) ***Path dependent options.*** *Let* $S = (S_t)_{0\le t\le T}$ *be a price process and assume that the price density* φ *is a function of* S_T*,* $\varphi = \varphi(S_T)$*, then*

$$C^* = g(S_T). \tag{28}$$

Thus any path dependent option $C = f(S)$ *can be improved by a European option*

$$C^* = g(S_T).$$

If φ *is increasing (decreasing), then* g *can be chosen decreasing (increasing). For this observation see Bernard et al. (2011b).*

d) ***Cost efficient options.*** *Given an option* X *with distribution function* F*, we consider the class* $\mathcal{C} = \mathcal{C}(F)$ *of all options which have the same payoff distribution as* X*,*

$$\mathcal{C} = \{C; F_C = F\} = \mathcal{C}(X). \tag{29}$$

As a corollary, Theorem 3.1 implies

Theorem 3.2 (Cost efficient claims) *For a given claim X and price density φ the claim*

$$C^* := F^{-1}(1 - \tau_\varphi(\varphi, V)) \in \mathcal{C}(X) \tag{30}$$

is a cost efficient claim, i.e.

$$E_Q C^* = \inf_{C \in \mathcal{C}(X)} E_Q C.$$

Proof. For the proof, note that any $C \in \mathcal{C}(X)$ satisfies that $C \leq_{\mathrm{cx}} X$. Thus Theorem 3.2 follows from Theorem 3.1. ∎

The notion of cost efficient claims was introduced in Dybvig (1988) and studied in the discrete case. It was extended in recent papers in Bernard and Boyle (2010) and Bernard et al. (2011b) to the case of continuous distributions. Several explicit results on lookback options, Asian options or related path dependent options in Black–Scholes type models are given in these papers.

3.1.2. Minimal demand problem

Closely related to the optimal investment problem is the minimal demand problem. Given a law invariant convex risk measure Ψ, a price measure $Q = \varphi P$ and a budget set

$$B = \{C; C \text{ claim}, E_Q C \leq c\}. \tag{31}$$

The *minimal demand problem* aims to find a claim C^* in the budget set with minimal risk

$$C^* \in B; \quad \Psi(C^*) = \inf\{\Psi(C); C \in B\}. \tag{32}$$

This problem has been discussed in Dana (2005), Schied (2004), and Föllmer and Schied (2004). An existence result is obtained in these papers for lsc convex risk measures. For law invariant convex risk measures, the Hoeffding–Fréchet bounds imply similarly as in Theorem 3.1.

Theorem 3.3 (Minimal demand problem) *There exists a solution C^* of the minimal demand problem* (32) *such that*

$$C^* = g(\varphi) \text{ for some } g \downarrow .$$

Remark 3.2 *For the corresponding utility maximization problem w.r.t. an expected utility function U*

$$U(C^*) = Eu(C^*) = \sup_{C \in B} U(C),$$

where u is a utility function. Explicit solutions are derived in He and Pearson (1991a,b) and many related papers for the standard utility functions. The solutions are obtained in the form

$$C^* = I(\lambda_Q(c))\varphi, \quad I(x) := (u')^{-1}(x), \tag{33}$$

where $\lambda_Q(x)$ is a constant chosen such that

$$E_Q C^* = c.$$

The main methods applied to solve this problem are a duality approach closely connected to a martingale approach (see Merton (1971), He and Pearson (1991a,b), and Kramkov and Schachermayer (1999)) and a projection approach based on φ-divergence distances (see Goll and Rü (2001) and Biagini and Frittelli (2005, 2008)).

3.2. Optimal (re-)insurance contracts

Optimal (re-)insurance contracts can be seen as particular instances of the optimal risk allocation problem. In this section we discuss some variations on the optimality of the classical stop-loss contracts which are obtained from the risk bound results for the comonotonic risk vector in Section 1, B1)–B3).

A (re-)insurance contract $I(X)$ for a risk $X \geq 0$ is defined by a function $I = \mathbb{R}_+ \to \mathbb{R}_+$, $0 \leq I(x) \leq x$, $I(0) = 0$. Let $\mathcal{I}$ denote the class of all reinsurance contracts (see Kaas et al. (2001)). $I \in \mathcal{I}$ is called an increasing insurance contract if $x - I(x)$ is increasing in x. If I is an increasing insurance contract, then I is 1-Lipschitz. The premium to be paid for the contract $I(X)$ is given by

$$\pi_I(X) = (1 + \vartheta)EI(X). \tag{34}$$

The stop-loss contract $I_d(X)$ – with retention limit d – is defined by

$$I_d(X) = (X - d)_+. \tag{35}$$

By a classical result going back to Arrow (1963, 1974) the stop-loss contract minimizes the retained risk $X - I(X)$ given a fixed premium π_0. The strongest version of this result is given in (Kaas et al. 2001, Example 10.4.4, p. 238).

Theorem 3.4 (Optimality of stop-loss contracts) *For any $I \in \mathcal{I}$ with $EI(X) = EI_d(X) = \frac{\pi_0}{1+\vartheta}$ with retained risks $R_I(X) := X - I(X)$ and $R_d(X) := X - I_d(X)$ holds*

$$R_d(X) \leq_{\mathrm{cx}} R_I(X). \tag{36}$$

Remark 3.3 *The proof in Kaas et al. (2001) is based on stochastic ordering and in particular on the Karlin–Novikov criterion for convex ordering. As a consequence of* (36)*, it holds for any law invariant convex risk measure Ψ that*

$$\Psi(R_d(X)) \leq \Psi(R_I(X)).$$

This result is reproved in Cheung et al. (2010a).

As in classical Markowitz theory we can formulate a corresponding efficient boundary result. Let Ψ be a law invariant convex risk measure and define for $I \in \mathcal{I}$

$$\begin{aligned} \mu_I &:= E(X - I(X)), & \sigma^2_\Psi(I) &:= \Psi(X - I(X)), \\ \mu(d) &:= \mu_{I_d}, & \sigma^2_\Psi(d) &:= \sigma^2_\Psi(I_d). \end{aligned}$$

Corollary 3.5 *Consider the risk set $\mathcal{R}_\Psi := \{(\mu_I, \sigma^2_\Psi(I)); I \in \mathcal{I}\}$ of all reinsurance contracts and the risk set $T_\Psi := \{(\mu(d)), \sigma^2_\Psi(d)); d \geq 0\}$ of all stop-loss contracts. Then the risk set T_Ψ of the stop-loss contracts is the lower boundary of the risk set $\mathcal{R}_\Psi$.*

Proof. The proof is similar as in the classical variance case. ∎

An interesting risk minimizing insurance protection problem for risks of joint portfolios was introduced in a recent paper of Cheung et al. (2010b). For a portfolio $X = \sum_{i=1}^n X_i$ with $X_i \sim F_i$, the risk is strongly influenced by the dependence of the components X_i of the joint portfolio. Let

$\mathcal{I}_n = \{I = (I_1, \dots, I_n); I_j$ increasing reinsurance contracts$\}$ denote the set of increasing reinsurance contracts of the joint portfolio. Let $\pi(I) = (1-\vartheta)\sum_{k=1}^n EI_k(X_k)$ denote the premium of contract I and π_0 be a given level of premium. $I^* \in \mathcal{I}_n$ is called the *optimal worst case reinsurance contract* if it solves the following problem:

$$R_\Psi(\pi_0) := \inf_{\substack{I\in\mathcal{I}_n\\ \pi(I)=\pi_0}} \sup_{X_i\sim F_i} \Psi\Big(\sum_{k=1}^n (X_k - I_k(X_k))\Big), \tag{37}$$

where Ψ is a law invariant convex risk measure. Thus with problem (37) one aims to find robust versions of reinsurance contracts which take into account the possible worst case dependence structure in the portfolio.

Cheung et al. (2010b) show in a recent paper that certain stop-loss contracts solve problem (37). This result can be obtained in a simplified way from the risk bound results in Section 1, which also allow to extend the result to general distributions not assuming continuity and strictly increasing distribution functions.

Theorem 3.6 (Optimal worst case reinsurance contracts) *The stop-loss contracts*

$$I_k^*(x) = I_{d_k^*}(x) = (x - d_k^*)_+, \quad 1 \le k \le n$$

as defined in (40) *are optimal worst case reinsurance contracts at premium* $\pi(I) = \pi_0$ *for any choice of law invariant convex risk measure* Ψ.

Proof. The proof follows from the risk bounds in Section 1 by the following two steps.

1) Since for $I \in \mathcal{I}_n : X_k - I_k(X_k) = (id - I_k)(X_k)$ is an increasing function of X_k, it follows from B1) and B3) that for any $X_k \sim F_k$ and $I \in \mathcal{I}_n$

$$\sum_{k=1}^n (X_k - I_k(X_k)) \le_{\mathrm{cx}} \sum_{k=1}^n (X_k^c - I_k(X_k^c)), \tag{38}$$

where $X^c = (X_k^c)$ is the comonotonic vector.

The comonotonic vector X^c is by (9) the worst case dependence structure. As a consequence of (38), we obtain

$$\Psi\left(\sum_{k=1}^n (X_k - I_k(X_k))\right) \le \Psi\left(\sum_{k=1}^n (X_k^c - I_k(X_k^c))\right) \tag{39}$$

for any law invariant convex risk measure Ψ (see (12)).

2) Let $d^* \ge 0$ satisfy $E(\sum_{k=1}^n X_k^c - d^*)_+ = \frac{\pi_0}{1+\vartheta}$, then there exist $d_k^* \ge 0$ such that $\sum_{k=1}^n d_k^* = d^*$ and

$$\left(\sum_{k=1}^n X_k^c - d^*\right)_+ = \sum_{k=1}^n (X_k^c - d_k^*)_+ \tag{40}$$

(see (11) for the choice of d_k^*).

For $I \in \mathcal{I}_n$ holds $\sum_{k=1}^n I_k(X_k^c) = J(X)$, where $X := \sum_{k=1}^n X_k^c$ and $J \in \mathcal{I}(X)$ is an increasing insurance contract. As a result, (39) and (40) together with the classical optimality result for stop-loss contracts in Theorem 3.4 imply optimality of I_k^*. ■

Remark 3.4 *Theorem 3.6 can be extended to the worst case risk problem with upper bounds on the premiums $\pi(I) \leq \pi_0$. This follows from the fact that $d_i^* = d_i^*(\pi_0)$ are increasing in π_0 (see* (11) *for the continuous case). For this result and examples see Cheung et al. (2010b).*

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CONTRIBUTED TALKS

PRICING OF SPREAD OPTIONS ON A BIVARIATE JUMP-MARKET AND STABILITY TO MODEL RISK

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1. INTRODUCTION

Recent considerations in finance have led to an increasing interest in multidimensional models with jumps taking the dependence between components into account (see for instance Cont and Tankov (2004)). In this context one is interested in finding closed-form formulas in such models for prices of options as spread options. A spread option is an option written on the difference of two underlying assets $S^{(2)}(t) - S^{(1)}(t)$, $t \geq 0$.

In the paper Benth et al. (2012) we consider a spread option of European type with strike 0 written in a bivariate jump-diffusion setting. Thus the pay-off function at maturity date T takes the form

$$\max(S^{(2)}(T) - S^{(1)}(T), 0),$$

where $(S^{(1)}(t), S^{(2)}(t))_{t\geq 0}$ is a bivariate jump-diffusion. We prove a Margrabe type formula for this spread option. The Margrabe formula is based on an appropriate change of measure which allows to move from pricing the spread option written in a bivariate process setting to pricing a European option written in a one-dimensional process setting (see Margrabe (1978) and Carmona and Durrleman (2003) for spread options written in continuous models). In our computations we use the Girsanov theorem to derive formulas for the spread option written in a bivariate jump-diffusion setting. We illustrate our results on spread options with several examples. We first compute spread option prices written in models with stochastic volatility. Moreover, we derive formulas for the spread option prices in the case the bivariate Lévy process has a NIG distribution and in the case of Merton dynamics. Therefore, we use the Girsanov theorem to describe the dynamics of the new.

Eberlein et al. (2009) studied the problem of valuation of options depending on several assets using a duality formula. In particular, they derived a formula for the valuation of spread options written in the setting of an exponential semimartingale described by the triplet of predictable characteristics of a one-dimensional semimartingale under the dual measure. In the paper Benth et al. (2012) we present a different approach for the valuation of spread options. Our approach is more direct and generalises their work to exponential jump diffusions with stochastic factors including stochastic volatility models.

From the modeling point of view, one can approximate the small jumps of the jump-diffusion by a continuous martingale appropriately scaled. This was introduced by Asmussen and Rosinski (2001) in the case of Lévy processes. Benth et al. (2010, 2011) studied convergence results of option prices written in one-dimensional jump-diffusion models. They also studied the robustness of the option prices after a change of measure where the measure depends on the model choice. In the paper Benth et al. (2012), we approximate the bivariate small jumps by a two-dimensional Brownian motion appropriately scaled and we prove the convergence of the spread option written on a bivariate jump-diffusion, modelling the two underlying assets. The main contribution in this paper is to apply the Margrabe type formula to prove the robustness of the spread option using one-dimensional Fourier techniques. We compute the convergence rate in the case the price process is driven by a bivariate Lévy process. Gaussian approximations of multivariate Lévy processes are studied in Cohen and Rosinski (2005).

In the present short paper we present the main results of the paper Benth et al. (2012). Namely, the Margrabe formula in a bivariate jump-diffusion framework and the robustness study of the spread options.

2. THE MARGRABE FORMULA IN A BIVARIATE JUMP-DIFFUSION FRAMEWORK

We first recall some basic results on Lévy processes and introduce the necessary notation. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space equipped with a filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ $(T > 0)$ satisfying the *usual conditions* (see Karatzas and Shreve (1991)). We introduce the generic notation $L = (L^{(1)}(t), \ldots, L^{(d)}(t))^*$, $0 \le t \le T$, for an $\mathbb{R}^d$-valued Lévy process on the given probability space. Here $()^*$ denotes the transpose of a given vector or a given matrix. We work with the right continuous version with left limits of the Lévy process and we let $\triangle L(t) := L(t) - L(t-)$. Denote the Lévy measure of L by $\nu(dz)$, satisfying

$$\int_{\mathbb{R}_0^d} \min(1, |z|^2)\, \nu(dz) < \infty,$$

where $|z| = \sqrt{\sum_{i=1}^d z_i^2}$ is the canonical norm in $\mathbb{R}^d$. Recall that $\nu(dz)$ is a σ-finite Borel measure on $\mathbb{R}_0^d := (\mathbb{R} - \{0\})^d$. From the Lévy-Itô decomposition of a Lévy process (see Sato (1999)), L can be written as

$$L(t) = at + \sigma^{\frac{1}{2}} B(t) + \int_0^t \int_{|z|\ge 1} z\, N(ds, dz) + \lim_{\varepsilon \downarrow 0} \int_0^t \int_{\varepsilon \le |z| < 1} z\, \widetilde{N}(ds, dz), \tag{1}$$

for a Brownian motion $B = (B^{(1)}(t), \ldots, B^{(d)}(t))^*$ in $\mathbb{R}^d$, a vector $a \in \mathbb{R}^d$ and a symmetric non-negative definite matrix $\sigma \in \mathbb{R}^{d\times d}$. $N(dt, dz) = N(dt, dz_1, \ldots, dz_d)$ is the Poisson random measure of L and $\widetilde{N}(dt,dz) := N(dt,dz) - \nu(dz)dt$ its compensated version. Notice here that $\int_0^t \int_{|z|\geq 1} z\, N(ds,dz) = \Big(\int_0^t \int_{|z|\geq 1} z_1\, N(ds,dz), \ldots, \int_0^t \int_{|z|\geq 1} z_d\, N(ds,dz) \Big)^*$. The convergence in (1) is $\mathbb{P}$-a.s. and uniform on bounded time intervals.

In the following, we consider a spread option of European type written on the difference of two underlying assets whose values are driven by a jump-diffusion. We consider a two-dimensional price process S given by the following dynamics under the measure $\mathbb{P}$:

$$dS(t) = S(t)\Big\{ a(t)dt + \sigma(t)dB(t) + \int_{\mathbb{R}_0^2} \gamma(t,z)\widetilde{N}(dt,dz) \Big\}, \tag{2}$$

where $a(t) = a(t,\omega) \in \mathbb{R}^2$, $\sigma(t) = \sigma(t,\omega) \in \mathbb{R}^{2\times 2}$, and $\gamma(t,z) = \gamma(t,z,\omega) \in \mathbb{R}^2$ are adapted processes. Note that the equation we consider for the price process is a stochastic differential equation using as integrators the Brownian motion B and the compensated compound Poisson process $\widetilde{N}$ of the Lévy process L defined in equation (1), where we choose $d = 2$.
The coefficients of the equation (2) are such that $\gamma_i(t, z_1, z_2) > -1$, $i = 1, 2$, for almost all $\omega \in \Omega$, $(t,z) \in [0,T) \times \mathbb{R}_0^2$, and moreover, for all $0 < t < T$, and $i = 1, 2$, we assume

$$\mathbb{E}\Big[\int_0^t \Big(|a_i(s)S^{(i)}(s)| + \sum_{j=1}^2 |\sigma_{ij}(s)S^{(i)}(s)|^2 + \int_{\mathbb{R}_0^2} |\gamma_i(s,z_1,z_2)S^{(i)}(s)|^2 \Big) ds \Big] < \infty, \quad \mathbb{P}\text{-a.s.} \tag{3}$$

The latter condition implies that the stochastic integrals are well defined and martingales.

Hereafter we detail the following Girsanov-type measure change, which will be useful in the sequel.

Lemma 2.1 *Define the measure $\widetilde{\mathbb{P}}$ by the Radon-Nikodym derivative with respect to $\mathbb{P}$ given on the σ-algebra $\mathcal{F}_T$ as follows*

$$\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}\bigg|_{\mathcal{F}_t} = \exp(Y(t)), \qquad 0 \leq t \leq T, \tag{4}$$

where

$$\begin{aligned} Y(t) = -\frac{1}{2}\int_0^t (\sigma_{11}^2(s) + \sigma_{12}^2(s))ds + \int_0^t \sigma_{11}(s)dB^{(1)}(s) + \int_0^t \sigma_{12}(s)dB^{(2)}(s) \\ + \int_0^t \int_{\mathbb{R}_0^2} \ln(1+\gamma_1(s,z_1,z_2)) - \gamma_1(s,z_1,z_2)\nu(dz_1,dz_2)ds \\ + \int_0^t \int_{\mathbb{R}_0^2} \ln(1+\gamma_1(s,z_1,z_2))\widetilde{N}(ds,dz_1,dz_2), \end{aligned} \tag{5}$$

satisfying

$$\mathbb{E}[\exp(Y(T))] = 1. \tag{6}$$

Thus the processes $B_{\widetilde{\mathbb{P}}}^{(1)}$ and $B_{\widetilde{\mathbb{P}}}^{(2)}$ defined by

$$dB_{\widetilde{\mathbb{P}}}^{(1)}(t) = -\sigma_{11}(t)dt + dB^{(1)}(t)$$

$$dB^{(2)}_{\widetilde{\mathbb{P}}}(t) = -\sigma_{12}(t)dt + dB^{(2)}(t)$$

are Brownian motions with respect to $\widetilde{\mathbb{P}}$ and

$$\widetilde{N}_{\widetilde{\mathbb{P}}}(dt, dz_1, dz_2) = -\gamma_1(t, z_1, z_2)\nu(dz_1, dz_2)dt + \widetilde{N}(dt, dz_1, dz_2) \tag{7}$$

is a compensated (time-inhomogeneous) Poisson random measure under $\widetilde{\mathbb{P}}$. We denote

$$\nu_{\widetilde{\mathbb{P}}}(dt, dz_1, dz_2) := -\gamma_1(t, z_1, z_2)\nu(dz_1, dz_2)dt$$

The spread is defined by the difference of the two underlying asset prices $S^{(2)}(t) - S^{(1)}(t)$, $t \geq 0$. Thus, the payout function of a European spread option with strike 0 at maturity date T is given by

$$\max(S^{(2)}(T) - S^{(1)}(T), 0) \ . \tag{8}$$

In the following we state a Margrabe type formula for a spread option written on a bivariate jump-diffusion (see Section 5.2 in Carmona and Durrleman (2003) for spread options written on continuous process prices). We choose the risk-free instantaneous interest rate $r(t) = r(t, \omega)$ to be an $\mathcal{F}_t$-adapted stochastic process which is Lebesgue integrable on any compact.

Proposition 2.2 *Assume that*

$$\exp\left(\int_0^T \{a_1(s) - r(s)\}ds\right) \max(\frac{S^{(2)}(T)}{S^{(1)}(T)} - 1, 0)$$

is $\widetilde{\mathbb{P}}$ integrable where the measure $\widetilde{\mathbb{P}}$ is defined in (4)*. Then the price C of a spread option with strike $K = 0$ and maturity T is given by*

$$C \ = S^{(1)}(0)\mathbb{E}_{\widetilde{\mathbb{P}}}\Big[\mathrm{e}^{\int_0^T \{a_1(s) - r(s)\}ds} \max(\tfrac{S^{(2)}(T)}{S^{(1)}(T)} - 1, 0)\Big] \ .$$

Notice that the price of $S^{(2)}$ expressed in the numéraire $S^{(1)}$ is a geometric jump diffusion. The dynamics under the measure $\widetilde{\mathbb{P}}$ can be easily computed.

3. ROBUSTNESS OF SPREAD OPTIONS

In this section we consider dynamics with no explicit Brownian component, namely we consider dynamics driven by a pure Lévy process or by centered Poisson random measures.

3.1. Robustness of the price process

Now we assume that the price process $S = (S^{(1)}, S^{(2)})$ is given by the following dynamics

$$S(t) \ = \ x + \int_0^t a(s)S(s)ds + \int_0^t \int_{\mathbb{R}_0^2} S(s)\gamma(s, z)\widetilde{N}(ds, dz), \tag{9}$$

where $S(0) = x \in \mathbb{R}^2$. We assume that the solution of the latter equation exists and that for $i = 1, 2$,

$$\gamma_i(s,z) = g_i(z)\widehat{\gamma}_i(s),$$

where $\int_{|z|\leq\varepsilon} g_i^2(z)\nu(dz) < \infty$. Moreover we assume that the stochastic factors $a_i(s)$ and $\widehat{\gamma}_i(s)$ are such that

$$|a_i(s)|, |\widehat{\gamma}_i(s)| \leq C, \qquad i = 1, 2,$$

where C is a positive constant (not depending on ω).

We define the matrix $G(\varepsilon) = \Big(G_{ij}(\varepsilon)\Big)_{1\leq i,j\leq 2}$, by

$$G_{ij}(\varepsilon) = \int_{|z|\leq\varepsilon} g_i(z)g_j(z)\nu(dz), \qquad \text{for } 1 \leq i, j \leq 2$$

and the matrix $\beta(\varepsilon)$ by the square root of $G(\varepsilon)$, namely

$$\beta(\varepsilon) = \begin{pmatrix} \beta_1(\varepsilon) & \beta_2(\varepsilon) \\ \beta_2(\varepsilon) & \beta_3(\varepsilon) \end{pmatrix} = G^{\frac{1}{2}}(\varepsilon). \tag{10}$$

We approximate the price process S by

$$\begin{aligned} S_\varepsilon^{(1)}(t) = x_1 &+ \int_0^t a_1(s)S_\varepsilon^{(1)}(s)ds + \beta_1(\varepsilon)\int_0^t S_\varepsilon^{(1)}(s)\widehat{\gamma}_1(s)dW^{(1)}(s) \\ &+ \beta_2(\varepsilon)\int_0^t S_\varepsilon^{(1)}(s)\widehat{\gamma}_1(s)dW^{(2)}(s) + \int_0^t\int_{|z|\geq\varepsilon} S_\varepsilon^{(1)}(s)\gamma_1(s,z)\widetilde{N}(ds,dz), \\ S_\varepsilon^{(2)}(t) = x_2 &+ \int_0^t a_2(s)S_\varepsilon^{(2)}(s)ds + \beta_2(\varepsilon)\int_0^t S_\varepsilon^{(2)}(s)\widehat{\gamma}_2(s)dW^{(1)}(s) \\ &+ \beta_3(\varepsilon)\int_0^t S_\varepsilon^{(2)}(s)\widehat{\gamma}_2(s)dW^{(2)}(s) + \int_0^t\int_{|z|\geq\varepsilon} S_\varepsilon^{(2)}(s)\gamma_2(s,z)\widetilde{N}(ds,dz), \end{aligned} \tag{11}$$

where $S_\varepsilon(0) = (x_1, x_2)$ and $W = (W^{(1)}, W^{(2)})$ is a two-dimensional Brownian motion. Notice here that the variance-covariance matrix of the process S_ε is given by $\widetilde{\Sigma}(\varepsilon,t) = \Big(\widetilde{\Sigma}_{i,j}(\varepsilon,t)\Big)_{1\leq i,j\leq 2}$, where

$$\begin{aligned} \widetilde{\Sigma}_{1,1}(\varepsilon,t) &= \big(\beta_1^2(\varepsilon) + \beta_2^2(\varepsilon)\big)\mathbb{E}\Big[\int_0^t (S_\varepsilon^{(1)}(s))^2\widehat{\gamma}_1^2(s)ds\Big], \\ \widetilde{\Sigma}_{1,2}(\varepsilon,t) &= \widetilde{\Sigma}_{2,1}(\varepsilon,t) = \big(\beta_1(\varepsilon)\beta_2(\varepsilon) + \beta_2(\varepsilon)\beta_3(\varepsilon)\big)\mathbb{E}\Big[\int_0^t S_\varepsilon^{(1)}(s)S_\varepsilon^{(2)}(s)\widehat{\gamma}_1(s)\widehat{\gamma}_2(s)ds\Big], \\ \widetilde{\Sigma}_{2,2}(\varepsilon,t) &= \big(\beta_2^2(\varepsilon) + \beta_3^2(\varepsilon)\big)\mathbb{E}\Big[\int_0^t (S_\varepsilon^{(2)}(s))^2\widehat{\gamma}_2^2(s)ds\Big]. \end{aligned}$$

Since the matrix $\beta(\varepsilon)$ is given by equation (10), the matrix $\widetilde{\Sigma}(\varepsilon)$ is the same as the variance-covariance matrix of the small jumps of the process S. We prove the following robustness result of the price process.

Proposition 3.1 *For every $0 \leq t \leq T < \infty$, we have*

$$\|S^{(1)}(t) - S_\varepsilon^{(1)}(t)\|_2^2 \leq CG_{11}(\varepsilon)\,,$$
$$\|S^{(2)}(t) - S_\varepsilon^{(2)}(t)\|_2^2 \leq CG_{22}(\varepsilon)\,,$$

where S and S_ε are solutions of (9) *and* (11)*, respectively and C is a positive constant depending on T, but independent of ε.*

3.2. Robustness of the Margrabe formula

In the following we study the robustness of the spread option written on a bivariate geometric Lévy process. We suppose that the dynamics of the price processes S and S^ε are given by equations (9) and (11), respectively. Applying Proposition 2.2, the price of the spread option written in the underlying process S is given by

$$C \; = S^{(1)}(0)\mathbb{E}_{\widetilde{\mathbb{P}}}\Big[e^{\int_0^T \{a_1(s)-r(s)ds\}} \max(\tfrac{S^{(2)}(T)}{S^{(1)}(T)} - 1, 0)\Big], \tag{12}$$

where the measure $\widetilde{\mathbb{P}}$ is defined by (4) for $\sigma_{11} = \sigma_{12} = 0$.

For the approximating processes, the spread option price is analogously given by

$$C_\varepsilon \; = S_\varepsilon^{(1)}(0)\mathbb{E}_{\widetilde{\mathbb{P}}_\varepsilon}\Big[e^{\int_0^T \{a_1(s)-r(s)\}ds} \max(\tfrac{S_\varepsilon^{(2)}(T)}{S_\varepsilon^{(1)}(T)} - 1, 0)\Big], \tag{13}$$

where $\widetilde{\mathbb{P}}_\varepsilon$ is defined by

$$\frac{d\widetilde{\mathbb{P}}_\varepsilon}{d\mathbb{P}}\bigg|_{\mathcal{F}_T} = \exp(Y_\varepsilon(T)).$$

Here above

$$\begin{aligned} Y_\varepsilon(T) = -\frac{1}{2}\Big(\beta_1^2(\varepsilon) + \beta_2^2(\varepsilon)\Big) \int_0^T \widehat{\gamma}_1^2(t)dt + \beta_1(\varepsilon) \int_0^T \widehat{\gamma}_1(t)dW^{(1)}(t) \\ + \beta_2(\varepsilon) \int_0^T \widehat{\gamma}_1^2(t)dW^{(2)}(t) \\ + \int_0^T \int_{|z|\geq\varepsilon} \ln(1+\gamma_1(t,z_1,z_2)) - \gamma_1(t,z_1,z_2)\nu(dz_1,dz_2)dt \\ + \int_0^T \int_{|z|\geq\varepsilon} \ln(1+\gamma_1(t,z_1,z_2))\widetilde{N}(dt,dz_1,dz_2). \end{aligned}$$

We can now conclude the following convergence result.

Proposition 3.2 *Let C and C_ε be defined in equations* (12) *and* (13)*. It holds that*

$$\lim_{\varepsilon \longrightarrow 0} C_\varepsilon = C.$$

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IMPLIED LIQUIDITY
AS ONE OF THE COMPONENTS OF MARKET FEAR

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Abstract

In this paper the concept of implied liquidity is discussed as one of the possible measures of the market fear. As the bid-ask spread can move in a constant market with no change in liquidity, this spread itself is not a perfect measure of liquidity. Hence the concept of implied liquidity is regarded in this paper as a proper measure that isolates and quantifies in a fundamental way liquidity risk in financial market. The idea of implied liquidity has its basis in recently developed two-way pricing theory (conic finance), where the traditional one-price model was replaced by a two-price model, yielding bid and ask prices for traded assets. Pricing is performed using distortion functions and distorted expectations. Calculations performed on the Dow Jones index show how liquidity dried out during the recent financial crisis. Intraday investigation shows a reasonable pattern of the liquidity parameter during a day.

1. INTRODUCTION

Market liquidity is regarded as one of the key measures in business, economics or investment, especially for risk-management purposes. It reflects an asset's ability to be sold. High bid-ask spreads characterize illiquid products, whereas liquidity implicates a smaller spread. However, it is very difficult to measure liquidity in an isolated manner. Bid-ask spreads can move around in a non-linear manner if spot or volatility moves, without a change in liquidity. In this paper we present the concept of implied liquidity as one of the possible measures of the market fear, which allows investors to measure the liquidity level of positions.

The concept of implied liquidity was proposed in (Corcuera et al. 2012). It is based on the fundamental theory of conic finance, in which the one-price theory is abandoned and replaced by a two-price theory yielding bid and ask prices for traded assets. For more background, we refer to Cherny and Madan (2009), Cherny and Madan (2010) and Madan and Schoutens (to appear, 2011). The pricing is performed by making use of non-linear distorted expectations employing a distortion function. In essence, the distorted expectation used in Cherny and Madan (2009) is parameterized by one parameter. A high value of this parameter gives rise to a wide bid-ask spread, a low value to a small spread. Given a market bid-ask spread, one can, via reverse engineering (cfr. implied volatility), back out the unique implied parameter to be put into the distortion function to recoup the market spread. This implied parameter is called the implied liquidity parameter. This allows us to measure the degree of liquidity of a certain asset in an isolated manner and to quantify it exactly.

Implied liquidity parameter is also used as one of the three components constituting the Fear Index FIX proposed in (Dhaene et al. 2011). FIX is created on the basis of the (equity) option surfaces on an index and its components. The quantification of the fear level is hence on the basis of option price data only and not on any kind of historical data. The index allows us to measure an overall level of fear in the market. The index takes into account market risk, via the VIX volatility barometer, liquidity risk, via the concept of implied liquidity, and finally systemic risk, via the concept of comonotonicity.

This paper is organized as follows. First we elaborate on the implied liquidity concept and present some basics of the conic finance theory. Subsequently distortion functions are explained and illustrated. Theory of bid-ask pricing is followed by the introduction of the implied liquidity parameter LIQ. Examples based on European Call options close the second section. In the next section the graphical representation of implied liquidity in the period of the recent financial crisis is shown. The chapter is closed with numerical results of current intraday observations of LIQ.

Computations are conducted on the basis of a historical study over the period January 2008 - October 2009, for which we calculated the implied liquidity level. Some key events in the recent credit crisis in that period are clearly identified. Intraday observations come from period 27.02.2012-2.03.2012. All calculations are done on the basis of Dow Jones index options data.

2. IMPLIED LIQUIDITY EXPLAINED

High bid-ask spreads characterize illiquid products, whereas liquidity implicates a smaller spread. However, it is very difficult to measure liquidity in an isolated manner. Bid-ask spreads can move around in a non-linear manner if spot or volatility moves, without a change in liquidity.

In the sequel, we will discuss the concept of implied liquidity, which in a unique and fundamentally founded way isolates and quantifies the liquidity risk in financial markets. The idea of implied liquidity has its basis in the recently developed two-way pricing theory (conic finance) proposed in Cherny and Madan (2009). In this theory, the one-price theory was replaced by a two-price theory, yielding bid and ask prices for traded assets. Pricing is performed by using distorted expectation with respect to the distortion function. We will start with summarizing the basics of the conic finance theory.

2.1. Conic finance - bid and ask pricing

In this section, we summarize the basic conic finance techniques needed to calculate the implied liquidity parameter related to a vanilla option position. For more background, see Cherny and

Madan (2009), Cherny and Madan (2010) and Madan and Schoutens (to appear, 2011). We will start with an introduction to distortion functions and distorted expectations.

2.1.1. Distortion functions and distorted expectations

Conic finance uses distortion functions to calculate distorted expectations. In Cherny and Madan (2009), a distortion function from the minmaxvar family parameterized by a single parameter $\lambda \geq 0$ as in Equation (1) is chosen.

$$\Phi(u;\lambda) = 1 - \left(1 - u^{\frac{1}{1+\lambda}}\right)^{1+\lambda} \tag{1}$$

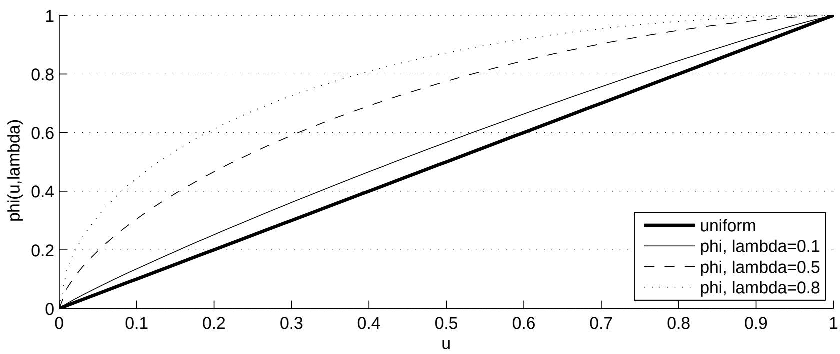

Figure 1: Distortion function

Figure 1 presents this distortion function for different values of parameter λ. One can observe, that larger values of λ give rise to a more concave $\Phi(u;\lambda)$ and that more probability is assigned to the down side values and less for the up side ones.

The distortion function is used to define a distorted expectation $de(X;\lambda)$. Operational cones were defined by Cherny and Madan (2009) and depend solely on the distribution function $G(x)$ of X and a distortion function Φ. Here a cashflow X is said to be acceptable, $X \in \mathcal{A}$, if the distorted expectation of X is non-negative. More precisely, the distorted expectation with respect to the distortion function Φ (we use the one given in Equation (1) but other distortion functions are also possible), of a random variable X with distribution function $G(x)$, is defined as

$$de(X;\lambda) = \int_{-\infty}^{+\infty} x \mathrm{d}\Phi(G(x);\lambda).$$

Note that if $\lambda = 0$, $\Phi(u;0) = u$ and hence $de(X;0) = \mathrm{E}[X]$ is equal to the original expectation.

One can consider the following example, which refers to Table 1. Let us consider 5 different payoffs: 0, 4, 7, 8 and 11. Assuming $\lambda = 0.8$, the following cumulative probabilities can be obtained:

cash flow	0	4	7	8	11
cumulative probability	0.2	0.4	0.6	0.8	1
distorted cumulative probability	0.6119	0.8087	0.9193	0.9791	1
probability	0.2	0.2	0.2	0.2	0.2
distorted probability	0.6119	0.1968	0.1106	0.0598	0.0209

Table 1: Distorted v.s. regular cumulative probability

In case of uniform probabilities, average cash flow equals to 6, whereas in case of the distorted case, the distorted average cash flow is 2.2697, which is much less than in regular case.

Hereafter, we will employ distorted expectations to calculate bid and ask prices.

2.1.2. Bid-Ask pricing

Cherny and Madan (2009) performed bid and ask price calculations using distortion functions and distorted expectations. The ask price of payoff X is determined by

$$ask(X) = -\exp(-rT)de(-X;\lambda).$$

This formula is derived by noting that the cash-flow of selling X at its ask price is acceptable in the relevant market, that is $ask(X) - X \in \mathcal{A}$. Similarly, the bid price of payoff X is determined as

$$bid(X) = \exp(-rT)de(X;\lambda)$$

Here the cash-flow of buying X at its bid price is acceptable in the relevant market : $X - bid(X) \in \mathcal{A}$.

One can prove that the bid and ask prices of a positive contingent claim X with distribution function $G(x)$ can be calculated as

$$bid(X) = \exp(-rT)\int_0^{+\infty} x\mathrm{d}\Phi(G(x);\lambda), \tag{2}$$

$$ask(X) = \exp(-rT)\int_{-\infty}^{0} (-x)\mathrm{d}\Phi(1-G(-x);\lambda). \tag{3}$$

Suppose now that we are given a market bid and ask price for a European call. We can then calculate the mid price of that call option, as the average of the bid and ask prices. Out of this mid price we calculate the implied Black-Scholes volatility. Next we can calculate the bid and ask prices (using the implied volatility as parameter). Under the Black-Scholes framework, this comes down to the following calculations for a European call option with strike K and maturity T. The distribution of the call payoff random variable to be used in (2) and (3) is in this case given by

$$G(x) = 1 - \mathrm{N}\left(\frac{\log(S_0/(K+x)) + (r-q-\sigma^2/2)T}{\sigma\sqrt{T}}\right), \qquad x \geq 0$$

where S_0 is the current stock price, r the risk-free rate and q the dividend yield. Further, N is the cumulative distribution function of the standard normal law and σ is the implied volatility determined on the basis of the mid price. For $x < 0$, $G(x) = 0$, since the payoff is a positive random variable. The above closed-form solution for $G(x)$ in combination with Equation (2) and (3) gives rise to very fast and accurate calculations of the bid and ask prices.

The following example shows how the bid-ask spread of European Call widens when the parameter λ increases. In this example the 25 days ATM European Call option on DJX index from 21.02.2012 was used. Closing stock price on that day was at the level of 129.66$ and mid price for strike 130$ was 1.385$. The implied Black-Scholes volatility was around 11% and we used an interest rate of 0.05%. In Figure 2 one can see the bid, mid and ask prices for the European Call option for a range of λ's.

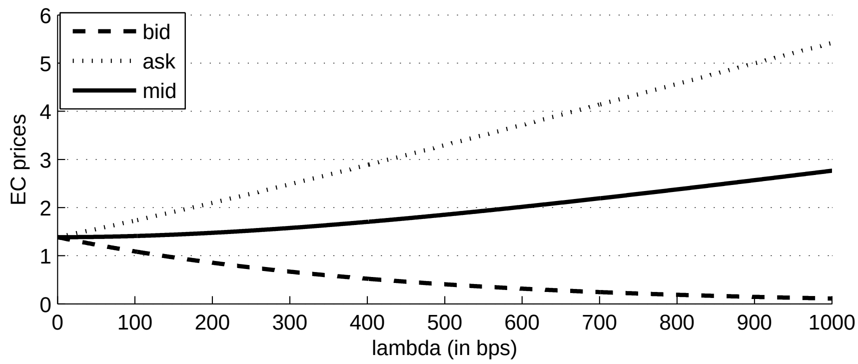

Figure 2: Bid, mid and ask prices for European Call options

One can observe that for $\lambda = 0$ we have the regular expectation and bid and ask prices equal the mid price of 1.385$. As λ increases, the bid-ask spread more and more widens.

The parameter λ in (2) and (3), for which the difference between the calculated and the market bid and ask prices is the smallest is called *the implied liquidity parameter*. The smaller the implied liquidity parameter, the more liquid the underlying and the smaller the bid-ask spread. In the extremal case where the implied liquidity parameter equals 0, the bid price coincides with the ask price, and we are back in the one-price framework. Results for the Dow Jones index are shown in the following section.

3. NUMERICAL RESULTS ON IMPLIED LIQUIDITY

It is well-known that a distressed market suffers from the drying up of liquidity. In the previous chapter we have overviewed the conic finance theory and pricing tools. Having this at hand, we come to a measure LIQ for quantifying liquidity risks in the market. This measure is based on the implied liquidity and measures the liquidity risk.

We denote by LIQ_j the 30–days implied liquidity of company j, calculated from the near and next term implied liquidities: $\lambda_j^*(T_1)$ and $\lambda_j^*(T_2)$. $\lambda_j^*(T_i), i = 1, 2$ itself is calculated as an average of all the individual implied liquidities of all non-zero bid call and put options of company j. Hence, LIQ_j of the j–th company is given by

$$LIQ_j = x_1 \lambda_j^*(T_1) + (1 - x_1)\lambda_j^*(T_2)$$

Here, x_1 is a weight calculated as:

$$x_1 = \frac{N_{T_2} - N_{30}}{N_{T_2} - N_{T_1}}$$

where:

- N_{T_1} = time to settlement of the near-term options (i.e. with maturity T_1);
- N_{T_2} = time to settlement of the next-term options (i.e. with maturity T_2);
- N_{30} = 30 days;

In the same way we calculate the implied liquidity LIQ_{DJX} of the index. This combination of near and next term liquidities provides a *short term forward looking implied liquidity*.

The overall liquidity index for a particular day is defined as:

$$\mathrm{LIQ} = \frac{1}{2}\mathrm{LIQ}_{DJX} + \frac{1}{2n}\sum_{j=1}^{n}\mathrm{LIQ}_j. \tag{4}$$

Equation 4 is further used for calculations of implied liquidity of Dow Jones index. LIQ was obtained for the crisis period and further we have conducted intraday observations of the implied liquidity over several days in February and March 2012. Results are presented in the following subsections.

3.1. Illustration during the crisis

In Figure 3 the market liquidity estimation based on the DJX index and all the 30 underlying stocks is presented. We clearly observe that LIQ is not constant over time and apparently exhibits a mean-reverting behavior. Recent work investigates this stochastic liquidity behavior more in depth, see (Corcuera et al. 2012).

The long run average of the implied liquidity of the data set in the period between January 2008 and October 2009 equals 390 bp. The highest value of the LIQ parameter, 1260 bp, was reached on the 24th of October 2008. Around this day several European banks were rescued by government intervention. Liquidity clearly dried up in these times of the high distress.

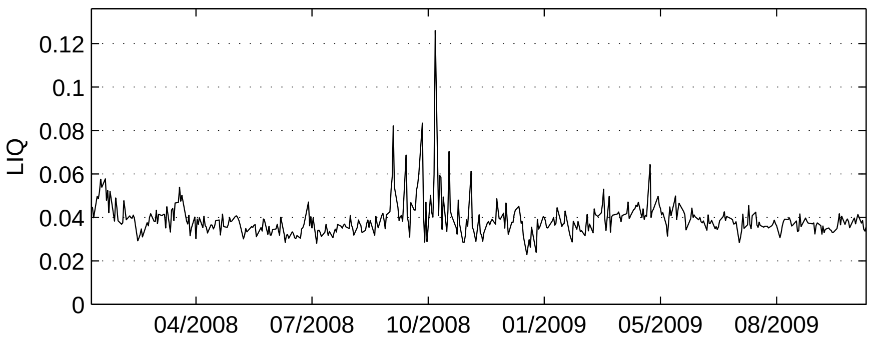

Figure 3: Implied liquidity, period 01.2008-10.2009

Current research focuses on intraday calculations, which are explained in the following section.

3.2. Intraday calculations

Historical data allowed us to calculate implied liquidity on a daily basis.

Figure 4 presents averaged weekly measurements of the implied liquidity during the day, for the time period 27.02.2012-2.03.2012. Calculations are done using the Dow Jones out-of-the-money options data.

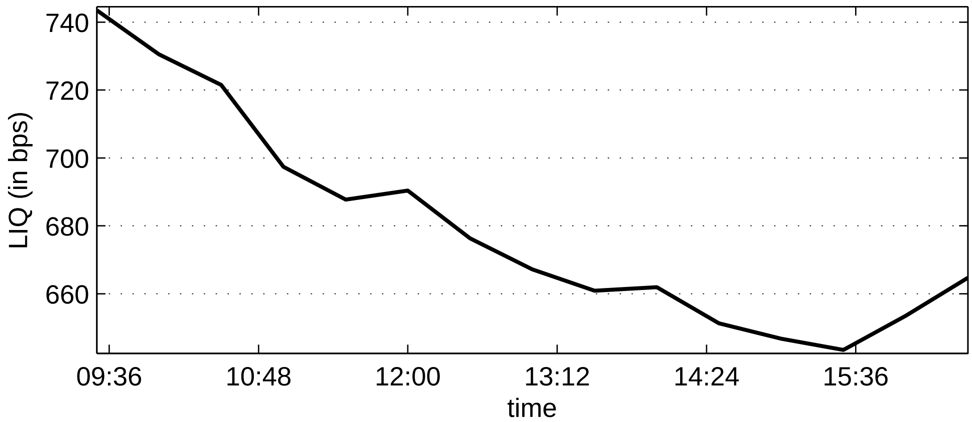

Figure 4: Average daily implied liquidity, period 27.02.2012-2.03.2012

We note that the U.S. stock market operates between 9:30am and 16:30pm. The pattern shows that a day usually starts with a high liquidity parameter value, which decreases during the day. This means that liquidity itself is on the lowest level at the start of a session and market becomes more and more liquid over the day. One can observe a small decrease of liquidity around lunch time and after that a further increase of liquidity. LIQ increases again in the end of the session, indicating drying-out liquidity at the end of the working day.

4. CONCLUSIONS

Since in a market the bid and ask spread can move around in a non-linear fashion with maturity and/or volatility and the spread can move in a constant market with no change in liquidity, spread itself is not a perfect measure of liquidity. In this paper the concept of implied liquidity measure is further discussed as a proper measure which isolates and quantifies in a fundamental way liquidity risk in financial markets.

Implied liquidity uses the concept of the conic finance, in which the one-price theory is abandoned and replaced by a two-price one, yielding bid and ask prices for traded assets. The pricing is performed by making use of non-linear distorted expectations.

Research is focused on the implied liquidity parameter LIQ intrinsically related to bid-ask spreads. We presented the historical values of the LIQ parameter solely based on vanilla index options and individual stock options of Dow Jones Index. Obtained results show that the liquidity clearly dried out in the time of the crisis in 2008. Moreover we observe that LIQ is not constant over time and exhibits a mean-reverting behavior.

The last section show results of the intraday calculations of implied liquidity, yielding the pattern of the liquidity parameter behavior.

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ON THE MARKET SELECTION HYPOTHESIS IN HETEROGENEOUS ECONOMIES

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Abstract

We analyze the market selection hypothesis introduced by Mielton Friedman (1953) in a fairly complex economy consisting of heterogeneous agents. We show that in the long run there exists a unique dominating agent whose consumption ratio converges to 1 as time evolves, whereas all other agents' consumption shares converge to 0. Furthermore, we prove that the equilibrium prices of the interest rate and the market price of risk are dictated by this sole dominating agent. Our main mathematical tool is Strassen's functional law of iterated logarithm along with some other probabilistic arguments and limit theorems for stochastic processes.

1. INTRODUCTION AND MAIN RESULTS

The market selection hypothesis (see Friedman (1953)) states that agents with a better assessment of the market, or with certain beneficial traits, are expected to govern the economy. On the other hand, irrational agents, or those ones with a rather superficial understanding of the market's dynamics are supposed to be eliminated from the economy in some sense. Translating the above ideas to more concrete statements, one can expect that "successful" agents are going to consume a considerable portion of the aggregate endowment in the economy, and have a significant impact on the market prices. In contrast to this, the "unsuccessful" agents will consume only a neglectable portion of the total endowment in economy (or even in a more radical scenario, their consumption shares will converge to zero, as time evolves to infinity), and will not affect the market prices at all.

Since the seminal work of Friedman (1953), the market selection hypothesis has attracted a wide attention especially in the last decade, when many researchers were aiming at establishing and examining the theoretical validity of this hypothesis in a variety of stochastic settings (see Blume and Easely (2006), Cvitanic et. al. (2011), Nishide and Rogers (2011), Yan (2008) and the references therein).

We examine the market selection in a highly heterogeneous and stochastic setting. We propose a model including simultaneously several sources of heterogeneity among the agents inhabiting the economy:

- Endowments.
- Impatience rates.
- Risk-aversions.
- Diverse beliefs.
- External habit-formation coefficients.

The first three notions are somewhat standard in the heterogeneous equilibrium literature. The latter two ingredients are relatively novel in the form presented here, since both of them are treated in parallel. We expand a bit our discussion on that: agents are allowed to have *diverse beliefs* concerning certain aspects of the market (the mean growth rate and the public signal delivering information). Furthermore, agents possess only partial information, i.e., some processes are unobservable (the mean growth rate) and are estimated by exploiting the available information (the aggregate endowment processes and a public signal). However, due to diverse beliefs, the filtered dynamics differ among agents and in particular differ from the actual dynamics under the physical measure. In terms of the preceding notions, the model we adopt here follows closely the works of Dumas et. al. (2009) and Scheinkman and Xiong (2003).

The preferences of the agents are modeled by *external habit-formation*. More specifically, the expected utility functions maximized by the agents are the so-called "catching up with the Joneses utilities"; see Chan and Kogan (2002) and Xiouros and Zapatero (2010). This is a model of habit-formation incorporating the impact of an external process called the standard of living, where each individual experiences a different level of this impact. The higher this level, the more sensitive the individual is to the patterns of the standard of living index. The closer is this index to 0, the closer are the preferences of the individual to "classical" CRRA (constant relative risk-aversion) utility functions. In our setting, the standard of living is defined as a geometric past average of the aggregate endowment process. Since we assume that the market clears (i.e., the aggregate endowment is equal to the aggregate consumption), the standard of living index can be interpreted as a process encoding the addiction of the individual caused by the past (aggregate) consumption.

Within the above framework, we prove (and provide an explicit characterization through a function called the survival index) the existence of a unique agent that dominates the market, as time growths to infinity. Furthermore, we show that in the long-run, the interest rate and the market price of risk derived endogenously in equilibrium are determined according to this dominating agent. Due to the fact that we can detect this surviving consumer by the characteristics of the agents, we conclude that the market selection hypothesis indeed holds true, at least in a modified form.

2. SETUP

2.1. Utility Maximization

The financial market in the model is assumed to be complete in the sense that it is represented by a (unique) positive state price density (or, pricing kernel) $(M_t)_{t\in[0,\infty)}$, and all claims adapted to the filtration $(\mathcal{G}_t)_{t\in[0,\infty)}$ (which is specified below) are dynamically hedgable. The market will be implemented later on by three assets (two risky and one riskless) with an interest rate and market price of risk determined endogenously by the market clearing condition leading to equilibrium. We start with a rather intuitive approach where we first present the utility maximization problem, despite that it is based on a certain structure arising from the fact that agents have diverse beliefs and use standard Kalman filtering to revise them. There are N agents in the economy ($i = 1, \ldots, N$), and each agent i solves the following expected **utility maximization** problem from consumption with an infinite time horizon:

$$\sup_{(c_t)_{t\in[0,\infty)}} E^{Q^i}\left[\int_0^\infty e^{-\rho_i t} U_i(c_t)dt\right],$$

under the budget constraint

$$E\left[\int_0^\infty c_{it} M_t dt\right] \leq E\left[\int_0^\infty \epsilon_{it} M_t dt\right].$$

As usual, ρ_i is the impatience rate, c_t denotes the consumption choice, and ϵ_{it} is the endowment process of agent i. The measure Q^i stands for a subjective probability measure (it will be set precisely in the sequel). Furthermore, the measure Q^i is equivalent to the physical measure P on the filtration $(\mathcal{G}_t)_{t\in[0,\infty)}$, but as we will explain below, they are not equivalent on a bigger filtration generated by all the shocks of the model. The subjective density (Radon-Nykodym derivative of these measures on the filtration $(\mathcal{G}_t)_{t\in[0,\infty)}$) is denoted by $Z_{it} := E\left[\frac{dQ^i}{dP}\big|\mathcal{G}_t\right]$, and an explicit formula is given below by some standard arguments from filtering theory combined with Girsanov's theorem. The utility function is of external habit-formation CRRA type and given by

$$U_i(c_t) = \frac{1}{1-\gamma_i}\left(\frac{c_t}{H_{it}}\right)^{1-\gamma_i},$$

where the process $(H_{it})_{t\in[0,\infty)}$ is called the "standard of living" index and given by

$$\log H_{it} = \beta_i e^{-\lambda t} \cdot \left(x_0 + \lambda \cdot \int_0^t e^{\lambda s} \cdot \log(D_s)ds\right),$$

where $(D_t)_{t\in[0,\infty)}$ is the aggregate endowment process, i.e.,

$$D_t := \sum_{i=1}^N \epsilon_{it}.$$

This specification of a habit-forming utility function postulates that the gratification attributed to a certain consumption process has to be discounted by the standard of living index. That is, there is a habit-formation affect (whose strength can change among agents) incorporated in the decision making procedure of each investor.

2.2. Equilibrium

It can be easily checked (due to the completeness of the market) that the optimal consumption stream of each agent i is given by

$$c_{it} = c_{i0} e^{\frac{\rho_i}{\gamma_i} t} M_t^{-\frac{1}{\gamma_i}} Z_{it}^{\frac{1}{\gamma_i}} H_{it}^{\frac{\gamma_i - 1}{\gamma_i}}.$$

We are now ready to introduce the standard notion of equilibrium in this setting.

Definition 2.1 $\big((c_{it})_{t\in[0,\infty)}, (M_t)_{t\in[0,\infty)}\big)$ *is an equilibrium, if:*
(a) It corresponds to the solution of the utility maximization problem of each agent i.
(b) The market clears:

$$\sum_{i=1}^{N} c_{it} = D_t.$$

The above definition combined with the explicit formula for the optimal consumption stream allows to provide an analytic description of the equilibrium state price densities.

Example. In a homogeneous economy type i (i.e., when the economy is represented by one agent of type i), the corresponding equilibrium state price density is given by

$$M_{it} = e^{-\rho_i t} D_t^{-\gamma_i} Z_{it} H_{it}^{(\gamma_i - 1)}.$$

Example. The heterogeneous (i.e. the general case) equilibrium state price density is given by

$$M_t = F(c_{10}^{\gamma_1} M_{1t}, \ldots, c_{N0}^{\gamma_N} M_{Nt}),$$

where $F(a_1, \ldots, a_N) : R_+^N \to R$ is defined via

$$\sum_{i=1}^{N} a_i^{1/\gamma_i} F^{-1/\gamma_i}(a_1, \ldots, a_N) = 1.$$

This is a very important formula as this allows to compute the dynamics of the state price density M_t by using Ito's formula and derive the corresponding dynamics of the interest rate and the market price of risk.

2.3. Endowment process and observability

Note that so far no specific assumptions were imposed on the dynamics of the underlying processes. We model the **aggregate endowment** process by the following dynamics

$$\frac{dD_t}{D_t} = \mu_t^D dt + \sigma^D dW_t^{(1)} \quad , \quad D_0 = 1.$$

Here, and henceforth, $W^{(j)}$, $j = 1, 2, 3$ are standard independent one-dimensional Wiener processes defined on the same probability space. This process is assumed to be *observable* by all agents. The constant σ^D is positive and the mean growth rate process μ_t^D is modeled by an mean-reverting process, i.e.

$$\mu_t^D = \overline{\mu} + (\mu_0 - \overline{\mu})\, e^{-\xi t} + \sigma^\mu e^{-\xi t} \int_0^t e^{\xi s} dW_s^{(2)},$$

where $\overline{\mu}, \mu_0, \xi > 0$. This process is assumed to be *unobservable*. There is another source of information available to all agents: this is an *observable public signal* given by

$$s_t = \phi W_t^{(2)} + \sqrt{1-\phi^2} W_t^{(3)},$$

where $\phi \in [0, 1)$. Thus we see that this public signal provides a certain valuable information to the agents since it exhibits a non-negative correlation (ϕ) with the shock governing the dynamics of the mean growth rate process. Consequently, agents will be aiming to optimally filter the (unobservable) dynamics of μ_t^D from the information conveyed by the aggregate endowment process and the public signal, i.e., their information set is the following sigma-algebra:

$$\mathcal{G}_t := \sigma\left(\{s_u; 0 \le u \le t\} \cup \{D_u; 0 \le u \le t\}\right).$$

2.4. Diverse Beliefs and Learning

Agents are allowed to have diverse beliefs concerning some aspects of the market. More specifically, each agent i beliefs that the *average* and *initial* growth rates differ from the correct ones and the correlation of the public signal with the mean-growth rate process is $\phi_i \in (-1, 1)$ and not the actual correlation ϕ. Mathematically, we denote by Q^i ($i = 1, \ldots, N$) a measure defined on the underlying probability space such that the latter dynamics are given by

$$\mu_t^D = \overline{\mu}_i + (\mu_{0i} - \overline{\mu}_i)\, e^{-\xi t} + \sigma^\mu e^{-\xi t} \int_0^t e^{\xi s} dW_s^{(2)}, \tag{1}$$

and

$$s_t = \phi_i W_t^{(2)} + \sqrt{1-\phi_i^2} W_t^{(3)}, \tag{2}$$

under this measure Q^i, where $W^{(i)}$, $i = 1, 2, 3$ are as before independent standard one-dimensional Wiener processes, under Q^i. Such measures always exist, given that the probability space is large enough. Note that (1) implies that each Q^i is a singular measure (with respect to the physical measure) on the large filtration generated by $W^{(1)}, W^{(2)}$ and $W^{(3)}$. We say that agent i is over-confident in the signal if $\phi > \phi$, otherwise we say that this agent is under-confident. Agents are of course in the process of learning the unobservable process μ_t^D by using their information set, but the filtering is executed under the subjective measure Q^i:

$$\mu_{it}^D = E^{Q_i}\left[\mu_t^D | \mathcal{G}_t\right].$$

Now, the dynamics can be derived explicitly by using the theory of optimal filtering:

$$\mu_{it}^D = \frac{\mu_{i0}}{y_{it}} + \frac{\xi \overline{\mu}_i}{y_{it}} \int_0^t y_{iu} du + \frac{1}{\left(\sigma^D\right)^2 y_{it}} \int_0^t \frac{\nu_{iu} y_{iu}}{D_u} dD_u + \frac{\sigma^\mu \phi_i}{y_{it}} \int_0^t y_{iu} ds_u, \tag{3}$$

where

$$y_{it} = \exp\left(\xi t + \frac{1}{(\sigma^D)^2}\int_0^t \nu_{is} ds\right), \tag{4}$$

and the variance process

$$\nu_{it} := E^{Q^i}\left[\left(\mu_t^D - E^{Q^i}\left[\mu_t^D | \mathcal{G}_t\right]\right)^2 | \mathcal{G}_t\right]$$

is deterministic and given by

$$\nu_{it} = \alpha_{i2}(\sigma^D)^2 \frac{e^{(\alpha_{i2}-\alpha_{i1})t} - 1}{e^{(\alpha_{i2}-\alpha_{i1})t} - \alpha_{i2}/\alpha_{i1}}, \tag{5}$$

where

$$\alpha_{i2} = \sqrt{\xi^2 + (\sigma^\mu/\sigma^D)^2(1-\phi_i^2)} - \xi,$$

and

$$\alpha_{i1} = -\sqrt{\xi^2 + (\sigma^\mu/\sigma^D)^2(1-\phi_i^2)} - \xi.$$

Now, we aim at showing that Q^i is in fact equivalent to the physical measure P on the smaller (information set) filtration $(\mathcal{G}_t)_{t\in[0,\infty)}$. For this purpose, denote by $i = 0$ a fictional rational agent who knows the correct dynamics of all the underlying processes, i.e., his or her subjective measure coincides with the physical measure P. Thus this agent filters the dynamics according to the following rule

$$\mu_{0t}^D = E^P\left[\mu_t^D | \mathcal{G}_t\right].$$

It is a well known fact that the process

$$dW_{0t} = dW_t^{(1)} - \frac{\mu_{0t}^D - \mu_t^D}{\sigma^D} dt$$

is a $(P, \mathcal{G})$ −Brownian motion. (In fact, it is evidently $\mathcal{G}$ adapted, and one can check that it is a Brownian motion by using Levy's theorem). We set next

$$\delta_{it} := \frac{\mu_{it}^D - \mu_{0t}^D}{\sigma^D}$$

to be $i-$th agent's **error** in estimation. Now, denote

$$dW_{it} = dW_t^{(1)} - \frac{\mu_{it}^D - \mu_t^D}{\sigma^D} dt$$

As above, it is a $(Q_i, \mathcal{G})$ −Brownian motion. On the other hand, we can write

$$dW_{it} = dW_{0t} - \delta_{it} dt.$$

Now, let P^i be a measure given by the following Radon-Nykodym derivative

$$Z_{it} := E\left[\frac{dP^i}{dP} | \mathcal{G}_t\right] = \exp\left(\int_0^t \delta_{is} dW_{0s} - \frac{1}{2}\int_0^t \delta_{is}^2 ds\right).$$

Then, it follows by Girsanov's theorem that W_{it} is a P^i-Brownian motion and also s_t is a Brownian motion. Now, we claim that the filtration $\mathcal{G}$ is generated by the public signal s_t and $W_t^{(i)}$, for arbitrary $i = 1, \dots, N$. To see this, note that

$$\frac{dD_t}{D_t} = \mu_{it}^D dt + \sigma^D dW_t^{(i)},$$

and

$$d\mu_{it}^D = -\xi \left(\mu_{it}^D - \overline{\mu}_i\right) dt + \frac{\nu_{it}}{\sigma^D} dW_t^{(i)} + \sigma^\mu \phi_i ds_t.$$

Finally, since s_t and $W^{(i)}$ are both Brownian motions under Q^i and P^i we get that $P^i = Q^i$ on the filtration $(\mathcal{G}_t)$, and we have in particular found the corresponding density process. From $i-$th agent's viewpoint, the dynamics of the total endowments process are given by

$$\frac{dD_t}{D_t} = \mu_{it}^D dt + \sigma^D dW_{it}.$$

3. MAIN RESULTS

For each agent i, we denote by

$$\kappa_i = \kappa(\beta_i, \gamma_i, \overline{\mu}_i, \phi_i, \rho_i) :=$$

$$\rho_i + \left(\overline{\mu} - \frac{1}{2}(\sigma^D)^2\right)(\gamma_i + (1-\gamma_i)\beta_i) +$$

$$\frac{1}{2}\left(\frac{\overline{\mu}_i - \overline{\mu}}{\sigma^D}\right)^2 + \frac{\xi^2 + \left(\sigma^\mu/\sigma^D\right)^2(1-\phi\phi_i)}{2\sqrt{\xi^2 + \left(\sigma^\mu/\sigma^D\right)^2(1-\phi_i^2)}}.$$

the survival index of this agent. As will be seen below this index depending on all agents' characteristics ranks the agents' surviving skills. The proofs of the results presented below can be found in Muraviev (2012).

Theorem 3.1 *Assume that there exists a unique agent I such that*

$$\kappa_I < \kappa_i,$$

for all $i \neq I$. Then, in equilibrium, the only surviving consumer in the long run is the one of type I:

$$\lim_{t\to\infty} \frac{c_{it}}{D_t} = \lim_{t\to\infty} \frac{c_{it}}{\sum_{j=1}^N c_{jt}} = 0,$$

for all $i \neq I$, and

$$\lim_{t\to\infty} \frac{c_{It}}{D_t} = \lim_{t\to\infty} \frac{c_{It}}{\sum_{j=1}^N c_{jt}} = 1.$$

Next, we would like to deal with asset prices. So far, we have assumed that there is a quite obscure market represented by the state price density derived explicitly in equilibrium. However, for getting results concerned with the long-run behavior of endogenous prices, we need to define the market structure. Recall that $\mathcal{G}_t = \sigma(s_t, W_{it})$, $i = 1, \ldots, N$. We implement the market by a bank account, one risky asset which pays a dividend being equal to the aggregate endowment process D_t, and one further asset that is not modeled explicitly. These prices are determined endogenously in equilibrium. Furthermore, it is a well known fact that in this case the *interest rate* and the *market price of risk* are detected through the dynamics of the state price densities (they are equal respectively to minus the *drift* and minus the *diffusion term*). We are now ready to state the following result.

Theorem 3.2 *(i) We have*

$$\lim_{t\to\infty} |r_t - r_{It}| = 0,$$

and

$$\lim_{t\to\infty} |\theta_t - \theta_{It}| = 0.$$

(ii) Under some conditions, we have

$$\limsup_{t\to\infty} |r_t - r_{it}| = +\infty,$$

and

$$\limsup_{t\to\infty} |\theta_t - \theta_{it}| = +\infty,$$

for all $i \neq I$.

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POSTER SESSION

HOW TO DEAL WITH UNOBSERVED HETEROGENEITY WHEN MODELLING BIVARIATE CLAIM COUNTS

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Bermúdez (2009) described bivariate Poisson regression models for ratemaking in car insurance. The central idea was that the existing dependence between the two different types of claims must be taken into account in order to achieve better ratemaking. However, another question comes up here. How to deal with the unobserved heterogeneity usually observed in such a data set when using a bivariate regression model. The aim of this paper is to show different bivariate claim counts models to account for such features of the data, i.e. overdispersion and excess of zeros. These models are applied to an automobile insurance claims data set with two different types of claims in order to analyse the consequences for pure and loaded premiums when the independence assumption is relaxed.

1. INTRODUCTION AND MOTIVATION

A priori ratemaking based on generalized linear models is usually accepted. Although it is possible to use the total number of claims as response variable, the nature of automobile insurance policies is such that the response variable is the number of claims for each type of guarantee. With the usual ratemaking procedure, a premium is obtained for each class of guarantee as a function of different factors. Then, assuming independence between types of claims, the total premium is obtained as the sum of the expected number of claims of each guarantee.

Two questions arise here:

1. Is the independence assumption realistic? When this assumption is relaxed, how might the tariff system be affected?
2. If the independence assumption is relaxed, how to deal with the unobserved heterogeneity usually observed in such a data?

In Bermúdez (2009), the bivariate Poisson regression models (BP) were presented as an instrument that can account for the underlying connection between two types of claims arising from the same policy. The conclusion was that even when there are small correlations between the counts, major differences in ratemaking may appear. Using BP models leads to a ratemaking that presents larger variances and, hence, larger loadings than those obtained under the independence assumption.

In automobile insurance, the problem of unobserved heterogeneity is caused by the differences in driving behaviour among policyholders that cannot be observed or measured by the actuary. The main consequence of unobserved heterogeneity is overdispersion. The presence of excess of zeros in most insurance data sets can be also seen as a consequence of unobserved heterogeneity.

To account for the excess of zeros, zero-inflated bivariate regression Poisson models (ZIBP) were included in the analysis in Bermúdez (2009). In contrast to the BP model, the marginal distributions of a ZIBP model are not of Poisson type and, as such, they can present overdispersion.

A natural way to allow for overdispersion is to consider mixtures of a simpler model. This is well done in the univariate setting when moving from the Poisson model to the negative binomial model. In Bermúdez and Karlis (2011), finite mixtures of BP regression models (FMBP) were considered.

2. BIVARIATE CLAIM COUNTS MODELS

2.1. Bivariate Poisson regression models

Let Y_1 and Y_2 be the number of claims for third-party liability and for the rest of guarantees respectively. It is assumed that Y_1 and Y_2 follow jointly a bivariate Poisson distribution (Kocherlakota and Kocherlakota (1992)):

$$(Y_1,\, Y_2) \sim BP(\lambda_1,\, \lambda_2,\, \lambda_3).$$

The BP distribution allows for positive dependence between Y_1 and Y_2; $\lambda_3 = Cov(Y_1,\, Y_2)$, is a measure of this dependence; finally, the marginal distributions are Poisson with $E[Y_k] = \lambda_k + \lambda_3$ for $k = 1, 2$.

If covariates are introduced to model λ_1, λ_2, and λ_3, the BP model can be defined by:

$$\log \lambda_{ki} = \beta_k^T \mathbf{x}_{ki},\; k = 1, 2, 3,\;\; i = 1, \dots, n$$

where $\mathbf{x}_{ki}$ is a vector of covariates for the i-th observation related to the k-th parameter and β_k is the associated vector of regression coefficients. For details, we refer to Bermúdez (2009).

In this model, the marginal means and variances are equal. Therefore, we need to consider extensions to allow for overdispersion.

2.2. Zero inflated bivariate Poisson regression models

From the above BP model, the ZIBP model is specified by:

$$f_{ZIBP}(Y_1, Y_2) = \begin{cases} (1-p) f_{BP}(Y_1, Y_2 | \lambda_1, \lambda_2, \lambda_3) + p f_D(Y_1 |\, \theta) & Y_1 \,=\, Y_2 \,=\, 0 \\ (1-p) f_{BP}(Y_1, Y_2 | \lambda_1, \lambda_2, \lambda_3) & \text{elsewhere,} \end{cases}$$

where $f_{BP}(Y_1, Y_2|\lambda_1, \lambda_2, \lambda_3)$ is the BP joint probability function, and $f_D(Y_1|\,\theta)$ is the degenerate probability function at zero. See Bermúdez (2009) for more details.

In contrast to the BP model, the marginal distributions are overdispersed. However, are these ZIBP models the best option to deal with the unobserved heterogeneity?

2.3. Finite mixture of bivariate Poisson regression models

In order to allow for overdispersion, mixtures of bivariate Poisson distribution can be considered starting by a $BP(a_1\lambda_1, a_2\lambda_2, a_3\lambda_3)$ distribution where the a_is jointly follow a trivariate distribution. The specification of the random-effects distribution can be a continuous, a discrete or a finite distribution. We consider the latter case giving rise to finite mixture models.

Namely, the FMBP model takes the form:

$$\begin{aligned}
\mathbf{Y}_i = (Y_{1i}, Y_{2i}) &\sim \sum_{j=1}^{m} p_j BP(y_1, y_2; \lambda_{1ji}, \lambda_{2ji}, \lambda_{3ji}), i = 1, \ldots, n,\ j = 1, \ldots, m, \\
\log(\lambda_{kji}) &= \beta_{kj}^T \mathbf{x}_{kji}, k = 1, 2, 3,\ j = 1, \ldots, m,
\end{aligned}$$

where $\mathbf{x}_{kji}$ is a vector of covariates for the i-th observation associated with the k-th parameter of the j-th component of the mixture and β_{kj} is the set of regression coefficients.

A natural extension of the model is to use covariates also in the mixing proportions, i.e. the vector of probabilities $(p_1, \ldots, p_m)$.

This model has some interesting features. First of all, the zero inflated model is a special case. Secondly, it allows for overdispersion, and thirdly, it allows for a neat interpretation based on the typical clustering usage of finite mixture models, see Bermúdez and Karlis (2011).

3. APPLICATION

A dataset containing information for 20,000 policyholders of the automobile portfolio of a major insurance company operating in Spain has been used. For each policy, 12 exogenous variables were considered plus the yearly number of accidents recorded for the two types of claim.

We have fitted models of increasing complexity to this data set, starting from a simple independent Poisson regression model (DP). In Table 1, it can be seen that the 2-FMBP regressions are by far the best models, especially the regression with covariates in the mixing proportion, which has the best AIC.

Model	Log-Lik	Parameters	AIC
Double Poisson	-48882.95	24	97813.90
Bivariate Poisson (BP)	-48135.98	25	96321.96
BP2 (regressors on λ_3)	-47873.37	26	95798.74
Zero inflated BP (ZIBP)	-45435.00	26	90922.00
ZIBP2 (regressors on λ_3)	-45414.80	27	90883.60
2-finite mixture BP (2-FMBP1)	-44927.01	51	89956.02
2-FMBP2 (regressors on p)	-44842.22	53	89737.44

Table 1: Information criteria for selecting the best model for the data

Five different, yet representative, profiles were selected from the portfolio to compare the impact of using these models in a priori ratemaking. The first can be classified as the best profile since it presents the lowest mean score. The second was chosen from among the profiles considered as good drivers, with a lower mean value than that of the average for the portfolio. A profile with a mean lying very close to this average was chosen for the third profile. Finally, a profile considered as being a bad driver (with a mean above the average) and the worst driver profile were selected.

Table 2 shows the results for the five profiles. The main differences in ratemaking when using bivariate models as opposed to the independent Poisson model is that bivariate models increase variances in most cases, meaning overdispersion. This is especially noticeable for ZIBP and FMBP models.

	Best		Good		Average		Bad		Worst	
Model	Mean	Var	Mean	Var	Mean	Var	Mean	Var	Mean	Var
DP	0.0793	0.0793	0.1070	0.1070	0.1866	0.1866	0.2860	0.2860	0.6969	0.6969
BP2	0.0873	0.1027	0.1131	0.1285	0.1804	0.1958	0.2824	0.3726	0.6920	0.7821
ZIBP2	0.0826	0.1037	0.1055	0.1371	0.1898	0.2822	0.2771	0.4963	0.5562	1.3440
FMBP2	0.0908	0.1514	0.0919	0.1430	0.2270	0.3787	0.3531	0.7482	0.5382	1.0184

Table 2: Comparision of a priori ratemaking

4. CONCLUSIONS

In Bermúdez (2009), ZIBP models were fitted to account for the excess of zeros found with respect to the simple BP models; at the same time, they allow for overdispersion. However, using FMBP models we show that in fact the problem is not merely zero inflation but more than this, so assuming the existence of two type of clients described separately by each component of the mixture significantly improves the modelling of the dataset.

In Bermúdez and Karlis (2011), a new model consisting of a finite mixture of bivariate Poisson regressions is proposed. The idea is that the data consist of subpopulations for which the regression structure is different. The model corrects for zero inflation and overdispersion.

The existence of “true” zeros assumed by ZIBP models may be a too strong assumption in some cases. However, the 2-FMBP model is not based on this somewhat strict assumption and allows mixing with respect to both zeros and positives. This idea is more flexible and it better holds in our case: the group separation is characterized by low mean with low variance for the first component ("good" drivers) and high mean with high variance for the second one ("bad" drivers).

Finally, most of the parameters show the same behavior for both "good" and "bad" drivers. However, we find three parameters which are only significant for the second component, so they can be used to define the "bad" drivers, basically parameters related to driver's age and the driving experience.

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USING WEIGHTED DISTRIBUTIONS TO MODEL OPERATIONAL RISK

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Abstract

The quantification of operational risk has, much more than other types of risk that banks and insurers are obliged to manage, to deal with various concerns regarding data. Several studies document some of those concerns. One of the main questions that worries both researchers and practitioners is the bias of data for the operational losses amounts recorded.

We support the assertions made by several authors and defend that the bias concern is a very serious problem when modeling operational losses data. The bias is presented in all databases, not only in the commercial databases provided by various vendors, but also in databases where the data for operational losses is collected and compiled internally.

We show that it's possible, based on mild assumptions on the internal procedures put in place to manage operational losses, to make parametric inference using loss data statistics. We estimate the parameters for the losses amounts, taking in consideration the bias that, not being considered, generates a twofold error in the estimators for the mean loss amount and the total loss amount, the former being overvalued and the last undervalued.

In this paper, we do not consider the existence of a threshold for which, all losses above, are reported and are available for analysis and estimation procedures. We follow a different approach to the parametric inference. We consider that the probability that a loss is reported and ends up recorded for analysis, increases with the size of the loss, what causes the bias in the database but, at the same time, we don't consider that a threshold exists, above which, all losses are recorded and available for analysis, hence, no loss has probability one of being recorded, in what we defend is a realist framework. We deduce general formulae, present some results for common theoretical distributions used to model (operational) losses amounts and estimate the impact for not considering the bias factor when estimating the value at risk.

Keywords: Operational Risk Management, Loss Data, Bias, Weighted Distributions, VaR.

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1. SAMPLING FRAME AND SAMPLE

Consider the following notation and hypotheses:

- X the individual operational loss amount is a random variable with cumulative distribution function $F_X(\cdot)$.
- $S_X = \{X_i, i = 1, \dots, N\}$ is the random sample of the operational losses occurred over a period, with the X_i independent and identically distributed (*i.i.d.*) with $F_X(\cdot)$.
- Not all the observations presented in the original sample S_X, will be available to model operational losses and for statistical inference, namely, parametric estimation. $S_Y = \{Y_j, j = 1, \dots, M\}$, with $M \leq N$ are the observations available for estimation sample). The unobservable S_X, produced by the original stochastic process, is called the sampling frame.
- Each individual loss presented in S_X has a probability, say p_i, $i = 1, \dots, N$, of being recorded and, in that case, belonging to the sample S_Y, the data that is available to us to study the phenomenon.

The researcher of operational losses ends up with a biased sample of all the operational losses that should have been reported. The bias is originated due to the positive correlation between the loss amount and the probability of being reported.

Each element in the sampling frame S_X, has probability of inclusion in the sample S_Y, depending on the quality of the mechanism put in place to filter the sampling frame and on the size of the element, with largest elements having bigger probabilities. If the mechanism is perfect, all the elements in the sampling frame would be selected and end up in the sample, so that we would have no loss of information and no biased sample.

After realization, the probability for an operational loss to be reported is dependent on the quality the mechanism put in place to record operational losses, and if the mechanism is not perfect, proportional to its likelihood.

2. WEIGHTED DISTRIBUTIONS

We can read the yearly work on weighted distributions in Fisher (1934). The problem of parameter estimation using non-equally probable sampling schemes was first addressed by Rao (1965), Patil and Rao (1977) and Patil and Rao (1978). In these papers the authors identified various sampling situations which can be modeled using weighted distributions and calculated the Fisher information for certain exponential families, focusing primarily on $w(x) = x$, for nonnegative random variables, denominating this weighted distributions by the *size-based form* of the original distribution.

Definition 2.1 *Assume a random variable X, with probability density function (pdf) (or probability mass function (pmf)) $f_\theta(x)$, with parameters θ in a given parameter space Θ. Also, assume that the values x and y are observed and recorded in the ratio of $w(x)/w(y)$, where $w(x)$ is a* *non-negative weight function, such that $\mathbb{E}(w(X))$ exists. If the relative probability that x will be observed and recorded is given by $w(x) \geq 0$, then the pdf of the observed data is*

$$f^w(x) = \frac{w(x)}{\omega} f(x), \text{where } w(x) \geq 0 \text{ and } \omega = \int_{\mathbb{R}} w(x) f_X(x) dx = \mathbb{E}(w(X)).$$

The pdf $f^w(x)$ is denominated the weighted pdf corresponding to $f(x)$.

Consider the following notation and hypotheses:

- N (and of course M) is a *rv*, although, depending on the sampling scheme used, the distribution of M conditional on N may be a degenerated random variable.
- The sample membership indicators $\mathbb{I}_k$, with $k = 1,\ldots, N$, are independent. The sampling scheme implies that the sampling is made without replacement. The sample membership indicators are distributed related to size according to

  $$\mathrm{P}(\mathbb{I}_k = 1 \mid X_k) = F_X^\xi(x_k),\ \xi \in [0, +\infty[.$$

  So, $\mathbb{I}_k \mid X_k \sim B\left(F_X^\xi(x)\right)$ has a Bernoulli distribution with probability of success $F_X^\xi(x)$. We can say that this is a particular case of a Poisson sampling design with inclusion probabilities proportional-to-size, see for instance Sarndal et al. (1992).
- ξ is as a censorship parameter (other possible analogies can be a disclosure or a quality parameter). If $\xi = 0$ (implying no censorship, total disclosure of all losses or a system so effective that all losses end up reported) we would have $\mathrm{P}(\mathbb{I}_k = 1 \mid X_k) = 1$, so that $S_Y = S_X$, and we would be in the usual situation of a random sample from $F_X(\cdot)$.

  However, when $\xi > 0$, we are in the presence of some degree of censorship in our sample, making more likely that big losses are included in the sample than small losses.

Proposition 2.1 *Let $X_1, \ldots, X_N$ be a random sample of individual losses, with X_i independent of N a random variable with support on $\mathbb{N}$. If we consider $S_X = \{X_1, \ldots, X_N\}$ as our sampling frame (or simply frame) and apply on S_X a sampling scheme proportional-to-size with no replacement, such that, P$(\mathbb{I}_i = 1 \mid X_i = x) = F_X^\xi(x)$, with $i = 1, \ldots, N$, where $F_X(\cdot)$ is the cdf of X_i and $\xi \in [0, +\infty[$ is the censorship parameter, then:*

a) Not conditional on knowledge of the frame, the inclusion variables are i.i.d. Bernoulli with $\pi = \frac{1}{\xi+1}$ the probability of success; $B\left(\frac{1}{\xi+1}\right) = B(\pi)$, that is,

$$P(\mathbb{I}_i = 1) = \frac{1}{\xi + 1} = \pi, i = 1, \ldots, N.$$

b) Since $\sharp S_Y = \sum_X \mathbb{I}_k = \sum_{i=1}^N \mathbb{I}_{X_i}$, we have that, $\mathbb{E}(\sharp S_Y \mid N) = N\pi = \frac{N}{\xi+1}$.

c) $P(X_j = x \mid \mathbb{I}_j = 1) = F_X^\xi(x) f_X(x)(\xi + 1)$, $j = 1, \ldots, N$, $\xi \in [0, +\infty[$.

Proposition 2.2 *With the assumptions of Proposition 2.1, the distribution of the observations in the sample, that is, the distribution of the losses recorded, hence, the distribution of the observations available to the researcher to make inference, is a weighted distribution on $f_X(\cdot)$ with weight function $w(x) = F_X^\xi(x)$.*

3. APPLICATION

We are only authorized to disclosure aggregated data. During 2010 the bank internal reports account for a total operational loss of $4\,414\,000$ €. This total loss was originated by $4\,700$ operations. So that we have a mean operational loss of 939 €. The risk department estimated a probability of $1/250$, for an operation to generate an operational loss and of 95% for the loss ending up reported and documented. In this case $\xi_0 = (1-95\%)/95\% = 0.05263$.

3.1. EXPONENTIAL MODEL

With the Exponential model, $f_X(x) = \frac{1}{\beta} e^{\left(-\frac{1}{\beta}x\right)} \mathbb{I}^+_{\mathbb{R}}(x), \beta > 0$,

$$f^w(x) = \left(1 - e^{-\frac{1}{\beta}x}\right)^{\xi} \frac{1}{\beta} e^{-\frac{1}{\beta}x} (\xi+1) \mathbb{I}_{\mathbb{R}^+}(x), \text{ with}$$

$$\mathbb{E}\left(X^w\right) = \int_{\mathbb{R}^+} x \left(1 - e^{-\frac{1}{\beta}x}\right)^{\xi} \frac{1}{\beta} e^{-\frac{1}{\beta}x} (\xi+1) dx = \beta H_{\xi+1},$$

where H_n is the n-th harmonic number. When we compare $\mathbb{E}\left(X^w\right)$ with $\mathbb{E}(X) = \beta$, we have that:

$$R = \frac{\mathbb{E}\left(X^w\right)}{\mathbb{E}(X)} = H_{\xi+1}.$$

For instance, when $\xi = 0.05263$, so that P$(\mathbb{I}_X = 1) = 95\%$, we have $\frac{\mathbb{E}(X^w)}{\mathbb{E}(X)} = H_{1.0263} = 1.03$, meaning, the expected value of a recorded loss is 3% larger than the original loss.
We have $\hat{\beta} = 939$ € *vs* $\hat{\beta}^w = 939/1.03 = 911.65$ €. The individual losses V@R1% is $F^{-1}_{\hat{\beta}}(99\%) = 4\,324.25$ € *vs* $F^{-1}_{\hat{\beta}^w}(99\%) = 4\,198.30$ €.
Estimating the true total operational losses the bank incurred, we have

$$\mathbb{E}\left(\sum_{i=1}^{N} X_i\right) = (1+\xi_0) \times 4\,700 \times 911.65 = 4\,510\,268.42 \text{ €}$$

estimating an increase of $96\,268.42$ € (2.18%).

3.2. PARETO MODEL

With the Pareto model, $f_x(x) = \frac{\alpha\beta^{\alpha}}{x^{\alpha+1}} \mathbb{I}_{]\beta,+\infty[}(x)$, with $\beta, \alpha \in \mathbb{R}^+$. We will consider the case with $\beta = 1$, but the generalization is straightforward, so

$$f^w(x) = \left(1 - x^{-\alpha}\right)^{\xi} \alpha x^{-(\alpha+1)} (\xi+1) \mathbb{I}_{]1,+\infty[(x)}, \text{ with}$$

$$\mathbb{E}\left(X^w\right) = (1+\xi) B\left(1 - 1/\alpha, 1+\xi\right)$$

where $B(x, y)$ is the beta function.
When we compare $\mathbb{E}(X^w)$ with $\mathbb{E}(X) = \alpha/(\alpha - 1)$, $\alpha > 1$, we have that:

$$R = \frac{\mathbb{E}(X^w)}{\mathbb{E}(X)} = (1+\xi)\frac{(\alpha-1)}{\alpha}B\left(1 - 1/\alpha, 1+\xi\right).$$

For instance, when $\xi = 0.05263$, so that $\mathrm{P}(\mathbb{I}_X = 1) = 95\%$, we have $\frac{\mathbb{E}(X^w)}{\mathbb{E}(X)} = 1.053$, meaning, the expected value of a recorded loss is 5.3% larger than the original loss.
We have, $\hat{\alpha} = 939/(939 - 1) = 1.0011$ € *vs* $\hat{\alpha}^w = 1.00112$ €. The individual losses V@R1% is $F_{\hat{\beta}}^{-1}(99\%) = 99.49$ € *vs* $F_{\hat{\beta}^w}^{-1}(99\%) = 99.49$ €.
Estimating the true total operational losses the bank incurred, we have

$$\mathbb{E}\left(\sum_{i=1}^{N} X_i\right) = (1+\xi_0) \times 4\,700 \times 893.86 = 4\,422\,254.74 \text{ €}$$

estimating an increase of $8\,254.74$ € (0.19%).

3.3. COMPARATIVE RESULTS

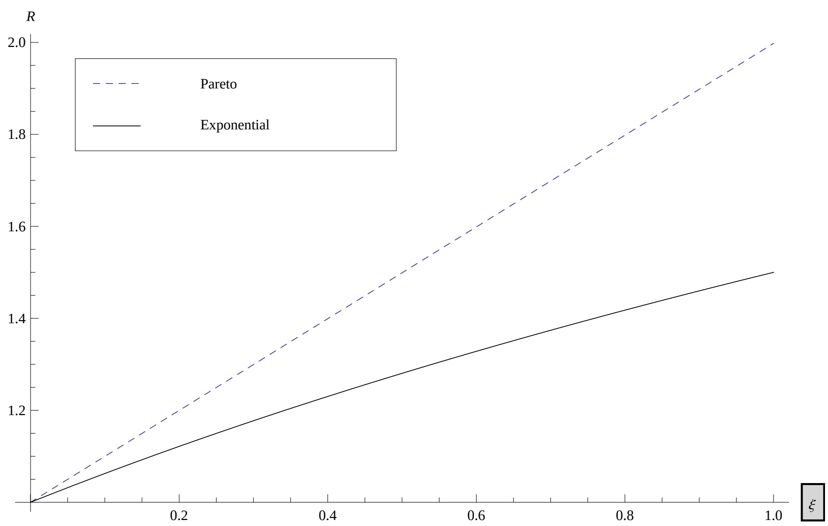

Figure 1: Ratio between the expected value of a recorded loss and the original loss.

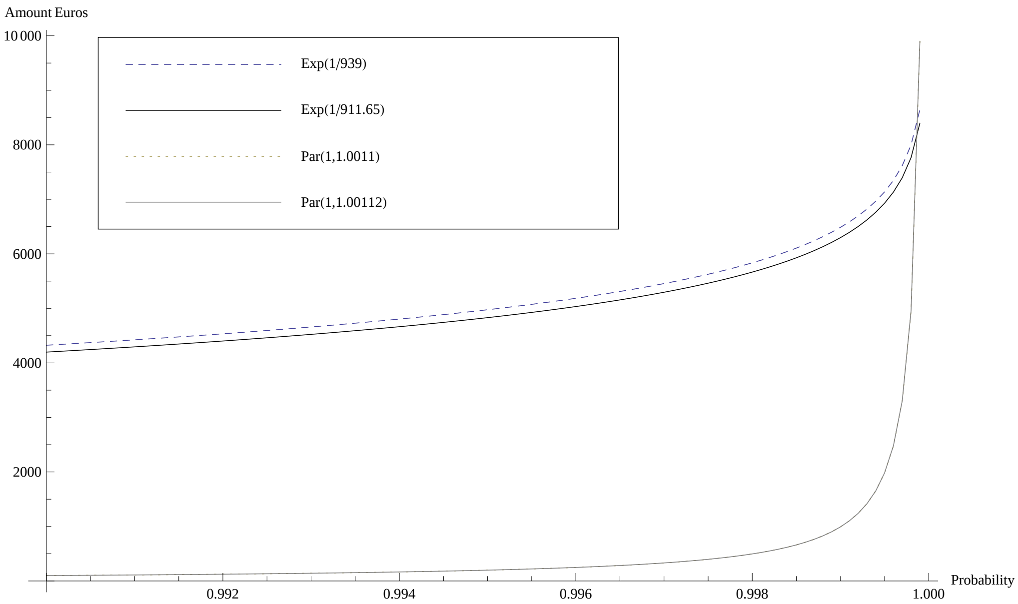

Figure 2: Quantiles 99% to 100% for Exponential and Pareto.

Figure 1 shows the relative increase in the expected value for $\xi \in [0, 1]$.

Figure 2 shows the difference in the quantiles (greater than 99%). If we consider an exponential model we have for a dashed line the individual losses reported and for the black line the true operational individual losses. For the Pareto model we don't have a substantial difference between the reported losses and the true losses.

4. CONCLUSIONS

What we learn from these two examples, considering the Exponential and Pareto distributions is that for a relatively high rate of success in reporting operational risk losses, 95% in our example, the heavy tail Pareto distribution is much less affected by the bias when estimating the parameters than the light tail Exponential distribution. Since our sampling scheme originates a bias towards the larger observations the parametric estimation is less affected in case of a right heavy tail distribution for the underlying data, as the observations that will not be considered have a bigger probability to be closer to values in the right tail. This can help to explain why the heavy tails are usually accepted as good (or not so bad) fits to operational risk loss data.

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BAYESIAN ESTIMATION OF THE CORRELATION MATRIX BETWEEN LINES OF BUSINESS FOR NON-LIFE UNDERWRITTING RISK MODULE SCR

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Based on Solvency II principles, the Quantitative Impact Study (QIS-5) provides a Standard Model to estimate an amount to ensure the stability against non expected adverse fluctuations, the solvency capital requirement (SRC). We focus on the non-life premium and reserves risk module. Non-life premium and reserves' SCR is mainly given by some parameters established by the CEIOPS. To apply the Standard Model, an insurance company can choose between using the standard parameters provided in QIS-5 or to estimate its own parameters based on its portfolio.

1. INTRODUCTION AND AIM

Our aim is to estimate the correlation matrix between lines of business given there is a lack of such estimator or methodology in QIS-5.

We propose the use of a Bayesian approach in order to fulfill this gap. We estimate the correlation matrix between lines of business mixing the information about correlations provided by the regulator and the one coming from the historical data through the use of a Bayesian model. The model is extensible to the correlation between premiums and reserve.

In the standard formula, the SCR corresponding to the risk premiums and reserves is calculated by means of a closed formula, which depends on a measure of volume, V, and an approximation of the mean-value-at-risk with a significance level of 99.5% at a one-year horizon, assuming a log-normal distribution of the underlying random variable, $\rho\left(\sigma\right)$:

$$SCR = V \cdot \rho\left(\sigma\right).$$

The expression $\rho\left(\sigma\right)$ depends on the combined standard deviation (σ) parameter. In the standard formula the combined standard deviation is obtained by, first, aggregating the corresponding standard deviation of premiums and standard deviation of reserves by lines of business, taking into account the existing correlation between them, thereby providing us with the standard deviation by line of business. Subsequently, by aggregating these and by taking into account the existing correlation between lines of business, the combined standard deviation is obtained.

The expression $\rho(\sigma)$, according to Gisler (2009), can be derived by considering a random variable, Z_i, which is the implicit random variable in the standard formula for the premium and reserve risk:

$$Z_i = \frac{X_i \cdot P_i + Y_i \cdot R_i}{V_i + R_i},$$

where X_i represents the loss ratio, P_i is a premium volume measure, Y_i represents the reserves ratio and P_i is a reserves volume measure, each time for the i-th line of business.

While Solvency II refers to the correlation matrix between lines of business, the approach here refers to the correlation matrix between the random variables Z_i for all pairs of lines of business. Different methodologies may be used to estimate this correlation matrix:

- Qualitative estimation:
  - Advantage: Stability over time.
  - Disadvantage: The estimations are subject to a high degree of subjectivity so the estimate is poor and suffers from a high degree of error.
- Quantitative estimation:
  - Advantage: Known estimators.
  - Disadvantage: Non-stability over time. The estimations are highly dependent on the number of observations and on its values.
- Credibility estimation:
  - Advantage: Take advantages of both qualitative and quantitative criteria.
  - Disadvantage: Needs to be hypothetized about variability of each one of the estimates, the qualitative and the quantitative.

In order to obtain a credibility formula, we propose to use a Bayesian approach allowing to find new correlation estimations mixing the information provided by the supervisor and the information obtained from the portfolio experience.

2. BAYESIAN MODEL

Following the notation stated in Lee (1998), the random variable considered here is the correlation coefficient ρ between two random variables X and Y (Z_i and Z_j for lines of business i and j respectively). According to Fisher (1915), using standard reference priors for μ_X, μ_Y, σ_X^2 and σ_Y^2, a reasonable approximation to the posterior probability of ρ is given by

$$p(\rho|x, y) \propto p(\rho) \cdot \frac{(1-\rho^2)^{\frac{n-1}{2}}}{(1-\rho\, r)^{n-\frac{3}{2}}}$$

where $p(\rho)$ is its prior density, x and y represent the random variables considered, r is the sample correlation coefficient between random variables x and y, and n is the number of observations used to calculate r.

Making the substitution $\rho = \tanh(\zeta)$ and $r = \tanh(z)$ and after another approximation, the random variable ζ follows a normal distribution with mean z and variance $\frac{1}{n}$, $\zeta \sim N\left(z, \frac{1}{n}\right)$. Assuming a normal prior distribution for z, we are into the situation of a normal prior and likelihood resulting in a normal posterior for z with mean

$$z_{\text{post}} = \frac{n_{\text{prior}}}{n_{\text{prior}} + n_{\text{likelihood}}} \times \tanh^{-1}(r_{\text{prior}}) + \frac{n_{\text{likelihood}}}{n_{\text{prior}} + n_{\text{likelihood}}} \times \tanh^{-1}(r_{\text{likelihood}})$$

and variance

$$\frac{1}{n_{\text{prior}} + n_{\text{likelihood}}}.$$

From z_{post}, we shall obtain a posterior point estimate for the coefficient correlation ρ by $r_{\text{post}} = \tanh(z_{\text{post}})$. The posterior estimate depends on the number of observations that we previously have, n_{prior}, and those incorporated from new information, $n_{\text{likelihood}}$; as well as of the previous estimate of the correlation coefficient, r_{prior}, and that obtained with the new information, $r_{\text{likelihood}}$.

3. APPLICATION

The data correspond to the historical aggregate volumes series for the period 2000 to 2010 of Spain's non-life market, see Ferri et al. (2012). Given the information available, only the first nine lines of business presented in the latest QIS have been considered. All the parameters necessary to obtain the SCR estimation, except the correlation matrix between lines of business, are those presented by the regulator as a *proxy*.

Table 1 shows the SCR estimation derived from three different methodologies for the estimation of the correlation matrix between lines of business, the qualitative one (QIS), the quantitative one (Empirical), and a credibility one (Bayesian) assuming several numbers of observations for the prior information, n_{prior}.

As can be seen, as higher numbers of observations for prior information are assumed, the SCR estimation tends to the value derived from the qualitative estimation of the correlation matrix between lines of business (6.66). This is due to the higher importance of the qualitative information through the credibility factor implicit in the Bayesian model.

Correlation Matrix between LoB	number of observations	SCR*
QIS		6.66
Empirical		6.45
Bayesian	$n_{\text{emp}} = 11$; $n_{\text{qis}} = 11$	6.48
	$n_{\text{emp}} = 11$; $n_{\text{qis}} = 50$	6.61
	$n_{\text{emp}} = 11$; $n_{\text{qis}} = 100$	6.64

* thousand million euro

Table 1: 2010 non-life underwriting SCR for Spanish aggregated market

4. DISCUSSION

The Bayesian Model examined here was proposed by Fisher (1915). In this model, it is assumed that prior and likelihood distributions are normally distributed, so the posterior estimate results in a normal distributed function with parameters, mean and variance, depending on the variability of the prior and the likelihood functions. In the case of the likelihood function, the variability can be easily determined since it comes from the number of observations of the historical data set. However, in order to determine the variability of the prior, we need to make an assumption on the number of observations needed by the regulator to form his judgment. It is clear that this assumption will determine the weights to obtain the posterior estimates of correlation coefficients. The merge of the criteria of the regulator and the criteria of insurers is a nice way to obtain estimates that allow certain flexibility. Nonetheless, further analysis on Bayesian models is needed in order to improve the estimations.

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DECIDING THE SALE OF THE LIFE INSURANCE POLICY IN CASE OF ILLNESS

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1. INTRODUCTION

Before the 80's, a policyholder who wanted to stop his life insurance contract had two options: to lapse the policy or to surrender the policy. The creation of the secondary market for life insurance policies gave a third option to the policyholder: to sell the contract to a third party for an amount greater than the cash surrender value (CSV) but lower than the policy face amount.
Secondary markets for life insurance policies started in the United States in the late 80's with the emergence of the viatical settlement. This product is only focused on insureds with terminally diseases that reduce their life expectancies to two years or less. During the 90's, the market expanded into another product, the life settlement, that allows impaired policyholders over 65 years and expected to die between the next two and fifteen years to sell the policy.
In both cases, the policy is sold to a viatical/life settlement provider for an amount that provides the seller with an immediate cash amount: the viatical settlement value (VSV) or the life settlement value (LSV). Then, the original policyholder transfers all the rights of the life policy and the provider shall pay all the remaining premiums and receives the death benefit when the insured dies. See Bhuyan (2009) or Aspinwall et al. (2009) for more information about viaticals and life settlement contracts.

In this extended abstract, we present two economic models within the framework of the utility theory that determine the optimal strategy for a policyholder who has to decide between to sell or not to sell his life policy in order to maximize his expected utility. The first model is focused on the viatical market and can be obtained in a discrete setting as the horizon planning is two years. The second model concerns a policyholder who wants to sell his policy in the life settlement market and should be treated in continuous time as the horizon planning is greater than two years. In both cases, we consider a decision maker of age x with a whole life insurance policy. Let M denote the death benefit and P the corresponding constant annual premiums. We denote by ${}_tp_x$ the probability that an individual age x is alive at age $x+t$ while ${}_{t/}q_x$ denotes the probability of a person age x dying between age $x+t$ and $x+t+1$. The cash surrender value of the whole life insurance policy at the beginning of year t is:

$$CSV_t = \phi \left[M \sum_{s=t}^{t_x-x-1} {}_{s/}q_x \cdot (1+r)^{-(s+1)} - P \sum_{s=t}^{t_x-x-1} {}_{s}p_x \cdot (1+r)^{-s} \right] > 0,$$

where $0 < \phi < 1$ and $t_x - x$ is the remaining lifetime for population of actual age x.

For the viatical model we assume that the decision maker has a maximum lifetime such that $\widehat{t_x} - x = 2$, where $\widehat{t_x}$ is the maximum age he can reach. For the life settlement model we consider he has a remaining life time such that $2 < \widehat{e_x} < 15$, where $\widehat{e_x}$ is the life expectancy. Henceforth the circumflex accent represents that probabilities have been adjusted. The insurance company does not take into account the illness of the insured while the viatical/life settlement provider does. Then, we can assume that:

$$VSV_t = \gamma_1 \left[M \sum_{s=t}^{1} {}_{s/}\widehat{q}_x \cdot (1+r)^{-(s+1)} - P \sum_{s=t}^{1} {}_{s}\widehat{p}_x \cdot (1+r)^{-s} \right], \qquad 0 < \gamma_1 < 1.$$

$$LSV_t = \gamma_2 \left[M \sum_{s=t}^{\widehat{t}_x-x-1} {}_{s/}\widehat{q}_x \cdot (1+r)^{-(s+1)} - P \sum_{s=t}^{\widehat{t}_x-x-1} {}_{s}\widehat{p}_x \cdot (1+r)^{-s} \right], \qquad 0 < \gamma_2 < 1.$$

where γ_1 and γ_2 are chosen such that $\gamma_1 > \gamma_2 > \phi$. Hence: $0 < CSV_t < LSV_t < VSV_t < M$. With respect to the utility functions, we consider:

$$\begin{aligned} U(C_i) &= \ln C_i, \\ V(H_{i+1}) &= \alpha \ln H_{i+1}, \ \alpha > 0, \end{aligned}$$

where $U(C_i)$ and $V(H_{i+1})$ are the utility functions with respect to consumption and bequest respectively and α indicates how the consumer values bequests in relation to consumption. The logarithmic utility functions are a particular case of potential utility functions. Logarithmic utility functions are also considered in Bhattacharya et al. (2004) and Yang (2012).

2. SELLING THE LIFE POLICY IN THE VIATICAL MARKET

At the beginning of the first period, $t = 0$, the decision maker has an initial wealth W and a whole life insurance policy. He consumes an amount $C_0 > 0$ during the first period and an amount $C_1 > 0$ during the second one. If he dies in the first period, he leaves to his beneficiaries an amount $H_1 > 0$ at $t = 1$, and if he survives the first period, then he dies for sure in the second period and leaves to his beneficiaries an amount $H_2 > 0$, at $t = 2$. The expected utility of the policyholder depends on the utilities of the consumption and the bequests for both periods that will be represented by $U(C_i)$ and $V(H_i)$; $i = 0, 1$. The objective of the policyholder who wants to sell his life policy in the viatical market is:

$$\max_{C_0, C_1} EU_0 = U(C_0) + \beta \cdot \widehat{q}_x \cdot V(H_1) + \beta \cdot \widehat{p}_x \cdot U(C_1) + \beta^2 \cdot \widehat{p}_x \cdot \widehat{q}_{x+1} \cdot V(H_2) \tag{1}$$

where $\beta \in (0, 1]$ is the yearly intertemporal discount factor. This individual presents a very high death probability for the first period $\widehat{q}_x$ (i.e., very low $\widehat{p}_x$). Should he survive the first period, then we know for certain that he will die in the second period, i.e., $\widehat{q}_{x+1} = 1$.

Assuming that the policyholder can sell parts of his life policy at the beginning of each year, we find five possible strategies. Viaticate $\delta\%$ at $t = 0$ and then at $t = 1$: viaticate the $(1 - \delta)\%$ (case 1) or viaticate the $\rho(1 - \delta)\%$ (case 2) or not viaticate (case 3). Not viaticate at $t = 0$ and then at $t = 1$: viaticate $\delta\%$ (case 4) or not viaticate (case 5).
The optimal strategy will be the one that maximizes the utility in (1) subject to the budget constraint derived from each case. The solution to the problem is found by using dynamic programming (see e.g. Bertsekas (2000)) and the Kuhn-Tucker technique. For each case, we first compute the optima at $t = 1$ and then update this optima to $t = 0$. The analytical results can be summarized as in the following.

For the case 1, 2 and 3:
Let

$$VSV_1^* = \begin{cases} VSV_1^{1-\delta} & \text{if case 1} \\ VSV_1^{\rho(1-\delta)} & \text{if case 2} \\ 0 & \text{if case 3} \end{cases} \qquad A^* = \begin{cases} 0 & \text{if case 1} \\ A'' & \text{if case 2} \\ A' & \text{if case 3} \end{cases} \qquad P^* = \begin{cases} 0 & \text{if case 1} \\ P'' & \text{if case 2} \\ P' & \text{if case 3} \end{cases}$$

and let

$$\overline{A} = \alpha\beta[(W + VSV_0^\delta - P' - C_0)(1 + r)^2 + (VSV_1^* - P^*)(1 + r)],$$

then:

- For $A^* < \overline{A}$, the solutions are

$$C_0^* = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

with:
$a = (1 + r)^2[1 + \alpha\beta q_x + \beta p_x \cdot (1 + \alpha\beta)]$,
$b = (-1)(1 + r)[(W + VSV_0^\delta - P^{'})(1 + r)(2 + \alpha\beta q_x + \beta p_x(1 + \alpha\beta)) + (VSV_1^* - P^*)(1 + \alpha\beta q_x) + A^{'}(1 + \beta p_x(1 + \alpha\beta)) + A^*(1 + r)^{-1}(1 + \alpha\beta q_x)]$,
$c = [(W + VSV_0^\delta - P^{'})(1 + r) + A'][(W + VSV_0^\delta - P^{'})(1 + r)(1 + r)^* + VSV_1^*(1 + r)^* - P^*(1 + r) + A^*]$,

$$H_1^* = (W + VSV_0^\delta - P' - C_0^*)(1 + r) + A',$$
$$C_1^* = \frac{(W + VSV_0^\delta - P' - C_0)(1 + r) + VSV_1^* - P^* + A^*(1 + r)^{-1}}{(1 + \alpha\beta)},$$
$$H_2^* = \frac{\alpha\beta}{1 + \alpha\beta}[(W + VSV_0^\delta - P' - C_0)(1 + r)^2 + (VSV_1^* - P^*)(1 + r) + A^*].$$

- For $A^* \geq \overline{A}$, the solutions are

$$C_0^* = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

with:

$a = (1+r)^2(1+\alpha\beta q_x + \beta p_x)$,
$b = (-1)(1+r)[(W + VSV_0^\delta - P^{'})(1+r)(2+\alpha\beta q_x + \beta p_x) + (VSV_1^* - P^*)(1+\alpha\beta q_x) + A^{'}(1+\beta p_x)]$,
$c = [(W + VSV_0^\delta - P^{'})(1+r) + A^{'}][(W + VSV_0^\delta - P^{'})(1+r) + VSV_1^* - P^*]$.

$$H_1^* = (W + VSV_0^\delta - P' - C_0^*)(1+r) + A',$$
$$C_1^* = (W + VSV_0^\delta - P' - C_0)(1+r) + VSV_1^* - P^*,$$
$$H_2^* = A^*.$$

In case $C_0^* \geq W + VSV_0^\delta - P' - P^*(1+r)^{-1}$, then $C_0^{*(2)} = W + VSV_0^\delta - P' - P^*(1+r)^{-1}$

For the case 4 and 5:
Let
$VSV_1^* = \begin{cases} VSV_1^\delta & \text{if case 4} \\ 0 & \text{if case 5} \end{cases}$ $\quad A^* = \begin{cases} A^{'} & \text{if case 4} \\ A & \text{if case 5} \end{cases}$ $\quad P^* = \begin{cases} P^{'} & \text{if case 4} \\ P & \text{if case 5} \end{cases}$
and let

$$\overline{A} = \alpha\beta[(W - P - C_0)(1+r)^2 + (VSV_1^* - P^*)(1+r)],$$

then:

- If $A^* < \overline{A}$, the optimal solutions are

$$C_0^* = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

with:
$a = (1+r)^2[1 + \alpha\beta q_x + \beta p_x(1+\alpha\beta)]$,
$b = (-1)(1+r)[(W-P)(1+r)(2+\alpha\beta q_x + \beta p_x(1+\alpha\beta)) + (VSV_1^* - P^*)(1+r)(1+\alpha\beta q_x) + A(1+\beta p_x(1+\alpha\beta)) + A^*(1+\alpha\beta \cdot q_x)(1+r)^{-1}]$,
$c = [(W-P) + A(1+r)^{-1}][(W-P)(1+r) + VSV_1^* - P^* + A^*(1+r)^{-1}]$.

$$H_1^* = (W - P - C_0^*)(1+r) + A,$$
$$C_1^* = \frac{(W - P - C_0)(1+r) + VSV_1^* - P^* + A^*(1+r)^{-1}}{(1+\alpha\beta)},$$
$$H_2^* = \frac{\alpha\beta}{1+\alpha\beta}[(W - P - C_0)(1+r)^2 + (VSV_1^* - P^*)(1+r) + A^*].$$

- If $A^* \geq \overline{A}$, the optimal solutions are

$$C_0^* = \frac{-b - \sqrt{b^2 - 4ac}}{2a},$$

with:
$a = (1+r)^2(1+\alpha\beta q_x + \beta p_x)$,
$b = (-1)(1+r)[(W-P)(1+r)(2+\alpha\beta q_x+\beta p_x) + (VSV_1^* - P^*)(1+r)(1+\alpha\beta q_x) + A(1+\beta p_x)$,
$c = [(W-P)(1+r)+A][(W-P)(1+r)+VSV_1^* - P^*]$.

$$\begin{aligned} H_1^* &= (W-P-C_0^*)(1+r)+A, \\ C_1^* &= (W-P-C_0)(1+r)+VSV_1^*-P^*, \\ H_2^* &= A^*. \end{aligned}$$

In case $C_0^* \geq W-P-P^*(1+r)^{-1}$, then $C_0^{*(2)} = W-P-P^*(1+r)^{-1}$.

By substituting the results in 1, we find all the EU_0 corresponding with each case. The optimal strategy will be the one that gives rise to a higher optimal value for EU_0.

3. SELLING THE LIFE POLICY IN THE LIFE SETTLEMENT MARKET

The objective of the policyholder who wants to sell his life policy in the life settlement market is:

$$\begin{aligned} \max\ EU_0 = E\,[\int_0^{s\wedge T} e^{-\rho j}\cdot U\left[C_1(j)\right]\ dj + e^{-\rho T}\cdot V\left[H_1(T)\right]\cdot 1_{\{T\leq s\}} \\ + \left[\int_s^T e^{-\rho u}\cdot U\left[C_2(u)\right]du + e^{-\rho T}\cdot V\left[H_2(T)\right]\right]\cdot 1_{\{T>s\}}\ ] \end{aligned}$$

The expectancy is due to the randomness of the moment of death T. After some arrangements, the problem of the uncertain lifetime can be simplified (Pliska and Ye (2007)):

$$\begin{aligned} \max\ EU_0 = \int_0^s \widehat{S}(j)\cdot e^{-\rho j}\cdot U[C_1(j)] + \widehat{f}(j)\cdot e^{-\rho j}\cdot V[H_1(j)]\ dj \\ + \int_s^{\widehat{t_x}-x} \widehat{S}_s(u)\cdot e^{-\rho u}\cdot U[C_2(u)] + \widehat{f}_s(u)\cdot e^{-\rho u}\cdot V[H_2(u)]\,du, \quad (2) \end{aligned}$$

where $\widehat{S}(t)$ is the survival function and $\widehat{f}(t)$ is the density function for an impaired insured.

To solve this optimization problem, the horizon planning is divided in two parts: $[0,s[$, i.e., the period before the sale of the policy, and $[s,\widehat{t_x}-x[$, i.e., the period after the sale of the life policy. For $t\in[0,s[$, the consumer generates utility for consumption $U[C_1(t)]$ and in case of death at $T<s$, there is utility from bequest $V[H_1(T)]$ where $H_1(T) = W_T + M$. For $t\in[s,\widehat{t_x}-x[$, the consumer generates utility for consumption $U[C_2(t)]$ and in case of death at $T>s$, there is utility from bequest $V[H_2(T)]$ where $H_2(T) = W_T + LSV_T$. At s, as the policyholder sells his policy, he receives the amount LSV_s and hence, there is a jump in the state variable W.

We consider also logarithmic utility functions. The optimal strategy corresponds with the maximum expected utility subject to the state equation $\dot{W} = Wr - C$ and assuming that $W(0) = W_0$. The solution to the problem is found by solving first the problem for $[s, \widehat{t_x} - x[$ using Dynamic Programmimg (Hamilton-Jacobi-Bellman equation). The value function obtained and updated at s is then replaced in (2) (instead of the second integral) as a final state for the problem $[0, s]$. This new problem is solved by using the Maximum Principle of Pontryagin.

The optimal consumption is:

$$c^*(t) = \frac{e^{-\rho t} \cdot \widehat{S}(t)}{e^{r(s-t)} \cdot \lambda^*(s) + \int_t^s e^{r(\tau-t)} \cdot e^{-\rho\tau} \cdot \widehat{f}(\tau) \cdot \beta \cdot \frac{1}{w(\tau)+M} \quad d\tau},$$

where $w^*(t)$ is the solution of the following integro differential equation:

$$\dot{w} = wr - \frac{e^{-\rho t} \cdot \widehat{S}(t)}{e^{r(s-t)} \cdot \lambda^*(s) + \int_t^s e^{r(\tau-t)} \cdot e^{-\rho\tau} \cdot \widehat{f}(\tau) \cdot \beta \cdot \frac{1}{w(\tau)+M} \quad d\tau}$$

where $\lambda^*(s) = e^{-\rho s} \cdot \widehat{S}(s) \cdot A^*(s) \cdot \frac{1}{w_s + LSV_s}$ and $A^*(s) = \int_s^{\widehat{t_x}-x} e^{-\rho\cdot(\tau-s)} \cdot (\widehat{S}(\tau) + \widehat{f}(\tau) \cdot \alpha) \;\, d\tau$.

For both models, we find the optimal solution (not selling or selling at some time unit) which will depend on some personal parameters of the agent and on the value paid for the policy, VSV or LSV. We also make a sensitivity analysis to see the influence on the optimal solution.

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PRICING OF GUARANTEED MINIMUM BENEFITS IN VARIABLE ANNUITIES

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The worldwide market of variable annuities (VAs) has been rapidly growing since their introduction in the mid-1980s in the United States. These fund-linked annuity products, which have become an essential part of the retirement plans in many countries, are often combined with additional living and death benefits. Since they are usually of a complex nature, consistent pricing of variable annuities becomes a difficult task. As there is often a tradeoff between a realistic model and analytical tractability, several studies in the literature either focus on closed-form solutions - by simplifying the contract setups and the modeling assumptions - or propose numerical methods for the multi-factor models. This work aims to fill this gap by showing how the explicit representations for prices of some of the VA products can be derived in a hybrid model for insurance and market risks.

1. INTRODUCTION

Over the years the guarantees provided on VA products have evolved as the market has adapted to meet customer needs. Depending on the benefit type, different GMxBs can now be seen on the market. Some of the most common examples include: Guaranteed Minimum Death Benefit (GMDB), Guaranteed Minimum Income Benefit (GMIB) and Guaranteed Minimum Withdrawal Benefit (GMWB). These products can differ in the way the guaranteed amount is determined. In some products only initial premiums are guaranteed, others guarantee all premiums paid plus accumulated interest (roll-up) or include the so-called ratchet options that raise the guaranteed amount depending on the underlying fund performance. We refer the reader to (Brunner and Krayzler 2009) for more information on different types of guarantees.

Since the introduction of variable annuities in the US, these products have been gaining special attention not only by practitioners, but also by researchers. Several papers had appeared over the last decade concerned with the pricing issues for different types of guaranteed minimum benefits embedded in variable annuities. Examples of these studies include the work of (Milevsky and Posner 2001) for GMDBs; (Milevsky and Salisbury 2006), (Dai et al. 2008) for GMWB contracts.

There seems, however, to be relatively little academic literature on GMIBs. A similar product, the so-called Guaranteed Annuity Option (GAO) was examined by, among others, (Boyle and Hardy 2003) and (Biffis and Millossovich 2006). GMIB in its actual form has been recently analyzed by (Marshall et al. 2010). Most of the papers dedicated to the pricing of variable annuities can be divided into two categories. The first group is interested in finding some analytical approximation and thus simplifying the models used or the products themselves. The others are focused on the numerical solutions within a more comprehensive and realistic pricing framework.

This paper aims to bridge this gap and provides a hybrid model (Hull-White-Black-Scholes with time-dependent volatility) within a general setup for pricing guarantees included in VAs. Furthermore, we extend the model to account for stochastic mortality. In the presented paper we focus on the Guaranteed Minimum Income Benefit and provide closed-form pricing formulae for this guarantee in the suggested hybrid model. Finally, we give an example of a GMIB contract, evaluate it in the proposed framework and analyze its price sensitivities with respect to the selected contract parameters.

2. VALUATION MODEL

In the following we extend the model of (Marshall et al. 2010) by introducing time-dependent volatility as well as stochastic mortality, independent from the financial market. Combined model is defined similar to the work of (Biffis and Millossovich 2006). Furthermore, we provide closed-form pricing formulae for Guaranteed Minimum Income Benefit in the presented framework.

Financial Market Model Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space. We assume the existence of an adapted short-rate process r as well as a risk-neutral pricing measure $\mathbb{Q}$. We describe the financial market under this measure via a Hull-White-Black-Scholes hybrid model with time-dependent volatility (HWBS^{tdv})[1]:

$$\begin{aligned} r(t) &= \phi(t) + x(t), \quad r(0) = r_0, \\ dx(t) &= -a_r x(t)dt + \sigma_r dW_r^{\mathbb{Q}}(t), \quad x(0) = 0, \\ dS(t) &= r(t)S(t)dt + \sigma_S(t)S(t)dW_S^{\mathbb{Q}}(t), \quad S(0) = S_0, \\ dW_S^{\mathbb{Q}}(t)dW_r^{\mathbb{Q}}(t) &= \rho_{Sr}dt. \end{aligned} \tag{1}$$

where a_r (mean reversion) and σ_r (volatility) are positive constants, $\phi(t), \sigma_S(t)$ are two deterministic functions, which can be calibrated to the term structures of interest rates and implied volatilities. Let $\mathbb{Q}^S$ denote the equity price measure, with equity price S used as a numeraire. Using the multi-dimensional version of Girsanov's theorem and rewriting the equity dynamics in terms of the log-return $Y = ln(S)$ we obtain under $\mathbb{Q}^S$:

$$\begin{aligned} dx(t) &= (-a_r x(t) + \sigma_r \sigma_S(t)\rho_{Sr})dt + \sigma_r dW_r^{\mathbb{Q}^S}(t), \\ dY(t) &= \left(r(t) + \frac{1}{2}\sigma_S^2(t) \right) dt + \sigma_S(t)dW_S^{\mathbb{Q}^S}(t). \end{aligned} \tag{2}$$

[1] We denote by x the stochastic part of the interest rate process and by S the equity price process.

After some calculations it can be shown that, conditional on the current filtration $\mathcal{F}_t$, both $x(T)$ and $Y(T)$ are normally distributed with some mean and variance denoted by $\mu_{x(T)}, \sigma_{x(T)}$ and $\mu_{Y(T)}, \sigma_{Y(T)}$ respectively.

Insurance Market Model We work on the same probability space and model a random lifetime of a person aged x at $t = 0$ as a stopping time $\tau(x)$ of a counting process $N_{x+t}(t)$ with corresponding mortality intensity $\lambda_{x+t}(t)$. We introduce two subfiltrations $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ and $\mathbb{H} = (\mathcal{H}_t)_{t\geq 0}$ of $\mathbb{F}$ by

$$\mathcal{G}_t = \sigma(\lambda_{x+s}(s) : s \leq t), \quad \mathcal{H}_t = \sigma(\mathbb{1}_{\{\tau(x)\leq s\}} : s \leq t).$$

In this setup, the **survival probability** can be defined as the probability that a person at the age of $x + t$ at time t survives at least up to time T:

$$p_{x+t}(t, T|\mathcal{G}_t) := \mathbb{P}(\tau(x) > T|\mathcal{G}_t \vee \mathcal{H}_t) = \mathbb{E}\left[e^{-\int_t^T \lambda_{x+s}(s)ds}|\mathcal{G}_t \vee \mathcal{H}_t\right].$$

Comparing the mortality intensity at time 0 with mortality intensity at time t, we introduce **mortality improvement ratio** as

$$\xi_{x+t}(t) = \frac{\lambda_{x+t}(t)}{\lambda_{x+t}(0)}.$$

Figure 1 gives an example of one simulated path of the mortality improvement ratio:

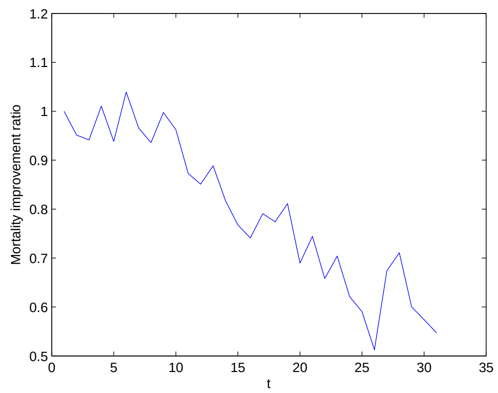

Figure 1: Mortality improvement ratio of a cohort aged 30 at time 0, $\xi_{30+t}(t)$.

Similar to (Dahl 2004) we use a two-step approach to model mortality intensity. In the first step we use a Gompertz model to describe the initial mortality intensity

$$\lambda_{x+t}(0) = bc^{x+t},$$

which can be calibrated to the current life table. In the second step we propose an extended Vasicek process to model $\xi(t)$:

$$d\xi(t) = k(e^{-\gamma_\xi t} - \xi(t))dt + \sigma_\xi dW_\xi(t).$$

The survival probabilities can then be expressed as

$$p_{x+t}(t, T|\mathcal{G}_t) = C_\lambda(t, T)e^{-D_\lambda(t,T)\lambda_{x+t}(t)}, \tag{3}$$

where $C_\lambda(t,T)$ and $D_\lambda(t,T)$ satisfy two ordinary differential equations, which can be solved analytically. Analogously to the financial market model, it can be shown, that mortality intensity at time T, $\lambda(T)$ [2] , is normally distributed with mean $\mu_{\lambda(T)}$ and variance $\sigma^2_{\lambda(T)}$.

3. PRICING

Guaranteed Minimum Income Benefit (GMIB) is a type of variable annuity that gives the policyholder an option at the retirement date T to obtain the account value $A(T)$ (without guarantee) or to annuitize the guaranteed amount $G(T)$ at some predefined annuitization rate g (the ratio between annual income and guaranteed amount). In other words, the policyholder can convert $A(T)$ into an annuity with fixed payments of $g \cdot G(T)$ at times $T_i > T$, $i = 1 \dots n$. The guaranteed amount $G(T)$ depends on the contract specification. In this work we consider a single premium GMIB with two common options for $G(T)$: *return of premium*, i.e. $G(T) = P$ and *roll-up*, i.e. $G(T) = Pe^{\delta T}$, where δ is the so-called roll-up rate. [3]

The value of GMIB at maturity T, conditioned on the policyholder survival until time T, can be written as:

$$V(T) = \mathbb{1}_{\{\tau > T\}} \max(A(T), G(T) \cdot g \cdot a_n(T)),$$

where $a_n(T)$ is the value at time T of an annuity paying one unit at times $T_1, \dots, T_n$. We assume that the policyholder's account is to 100% invested in the underlying equity fund and, thus, has the same dynamics as S with initial value equal to the single premium P, i.e.

$$dA(t) = A(t)\frac{dS(t)}{S(t)}, \quad A(0) = P.$$

The time 0 fair value of GMIB can be written as [4]

$$V(0) = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_0^T r(s)ds}\mathbb{1}_{\{\tau > T\}} \max(A(T), G(T) \cdot g \cdot a_n(T))\right]. \tag{4}$$

Theorem 3.1 *The price at time* 0 *of the GMIB for a person at the age of* x *is*

$$V(0) = PC_\lambda(0,T)e^{-D_\lambda(0,T)\lambda(0)}\left(1 + e^{\delta T} g \sum_{i=1}^{n} \left[F_i N(h_i^1) - K_i N(h_i^2)\right]\right)$$

[2] From now on, for convenience, we skip the subindex denoting the person's age.
[3] It should be mentioned that the first option is a particular case of the second one, for $\delta = 0$.
[4] In the following we omit the initial filtration $\mathcal{F}_0$.

where

$$
\begin{aligned}
F_i &= e^{M_i+\frac{1}{2}V_i}, \\
h_i^1 &= \frac{\ln\left(\frac{F_i}{K_i}\right)+\frac{1}{2}V_i}{\sqrt{V_i}}, \\
h_i^2 &= h_i^1-\sqrt{V_i}, \\
M_i &= \ln(\tilde{C}_i)=\ln\big(P^{-1}C_\lambda(T,T_i)C_r(T,T_i)e^{-D_\lambda(T,T_i)\mu_{\lambda(T)}} \\
&\quad \times e^{-D_x(T,T_i)\mu_{x(T)}-\mu_{Y(T)}}\big), \\
V_i &= \tilde{D}_i^2 = D_\lambda^2(T,T_i)\sigma^2_{\lambda(T)}+D_x^2(T,T_i)\sigma^2_{x(T)}+\sigma^2_{Y(T)} \\
&\quad +2D_x(T,T_i)\sigma_{x(T)}\sigma_{Y(T)}\rho_{x(T),Y(T)}
\end{aligned}
$$

K_i *is defined as* $K_i := \tilde{C}_i e^{-\tilde{D}_i z^*}$, *where* z^* *is a solution of*

$$\sum_{i=1}^{n}\tilde{C}_i e^{-\tilde{D}_i z^*}=K, \quad K=(ge^{\delta T})^{-1}.$$

4. EXAMPLE

In this part we specify a GMIB product and show its price sensitivities to different annuity maturities and different predefined annuitization rates (see Figure 2). We consider the following VA contract:

- Type of the guarantee: single premium GMIB
- Guaranteed annuitization rate g: 7.5%; roll-up rate δ: 2%
- Maturity of the guarantee T: 10 years; of the annuity n: 20 years, annual payments
- Policyholder: male, 55 year old

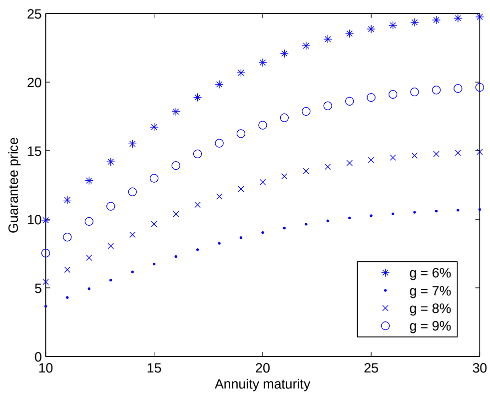

Figure 2: GMIB prices for different annuitization rates and maturities. Source: own calculations, models calibrated to the data from Federal Statistical Office of Germany and Bloomberg.

5. CONCLUSION

In this work we presented a hybrid model for financial and insurance markets in which closed-form formulae for the pricing of Guaranteed Minimum Income Benefits are derived. This work can be seen as an extension to some widespread models in the literature, e.g. inclusion of stochastic interest rates and stochastic mortality compared to (Bauer et al. 2008) and (Milevsky and Salisbury 2006), time-dependent volatility as well as explicit incorporation of mortality modeling in the framework of (Marshall et al. 2010). Furthermore, as opposed to several papers on the valuation of equity-linked products, where numerical pricing of the guarantees is suggested, analytical closed-form expressions are provided in the presented framework. In the next steps an extension of the 2-factor model for interest rates, incorporation of policyholder behavior risk as well as the analysis of further guarantees should deserve special attention.

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OPTIMAL ASSET-ALLOCATION WITH MACROECONOMIC CONDITIONS AND LABOR INCOME UNCERTAINTY

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This paper analyzes optimal asset-allocation strategies over the life cycle of an individual by taking into account both the labor income risk profile of the individual, and the macroeconomic dynamics of the assets included in the portfolio. Changes of the states of the economy are characterized by a discrete time, discrete space Markov chain aiming to capture the dynamics of the market (expansion and recession states). Since the dynamics of the financial market affects the labor income of the individual, labor income risk is characterized by a correlation level between the market and the labor income process. Under this model, optimal asset-allocation strategies are characterized.

1. INTRODUCTION

This paper analyzes the optimal asset-allocation over the life cycle of an individual when his labor income is uncertain during his working life period due to the nature of his profession or due to the economic prevailing conditions. Workers with same individual characteristics such as age, education level, geographical location and same labor income on average, still may be exposed to very different labor income risks: individuals with safe jobs are less sensitive to market movements and macroeconomic conditions, while individuals with uncertain wages are much more sensitive to them. There exists a huge literature upon optimal asset-allocation over the life cycle[1]. The seminal work of Merton (1969) analyzes optimal asset-allocation under uncertainty by assuming that labor income follows a stochastic process that is perfectly correlated to the stock process. If the individual is allowed to take short positions, the individual can hedge his labor income risk with the available financial assets. Bodie et al. (1992) extend the optimal porfolio analysis by including labor flexibility, that is, the individual faces the decision about the amount of hours he allocates to work and to leisure. El Karoui and Jeanblanc-Picqué (1998) and Koo (1999) derive properties of

[1]See Bodie et al. (2009) for a review of the recent scientific literature.

optimal asset-allocation by incorporating the fact that individuals can not borrow against future labor income, so liquidity constraints are imposed into the model. Henderson (2005) studies optimal asset-allocation when market and labor income are imperfectly correlated. The contribution of this paper is to characterize optimal asset-allocation over the life cycle of an individual that is liquid constrained while incorporating both the correlation of the labor income with the market, and the change of states or macroeconomic conditions on the market (regime-switching). The next section describes the model, Section 3 states the optimal asset-allocation problem, Section 4 provides the numerical implementation and results, while Section 5 concludes.

2. THE MODEL

The states of the economy. Assume that the economy activities take place on discrete time instants. Let $\mathcal{T}$ be the time index set $\{0, 1, 2, \ldots, T\}$, where $T < \infty$. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and $\mathbf{m} = \{\mathbf{m_t} \,|t \in \mathcal{T}\}$ be a discrete time finite-state hidden Markov chain on $(\Omega, \mathcal{F}, P)$ that describes the evolution of the different states of the economy $\mathbf{s} = \{\mathbf{s_1}, \mathbf{s_2}, \ldots, \mathbf{s_K}\}$. The states are identified by standard unit vectors $\{\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_K}\}$, where $\mathbf{e_i} = (0, \ldots, 1, \ldots, 0)'$ and $'$ denotes the transpose. Assume the Markov chain $\mathbf{m}$ is time homogeneous. Let $p_{i,j} = \Pr(\mathbf{m_{t+1}} = \mathbf{e_j} \,|\mathbf{m_t} = \mathbf{e_i})$ for $i, j = 1, \ldots, K$ be the transition probabilities and denote by $\mathbf{A}$ the transition probability matrix $[p_{i,j}]_{i,j=1,\ldots,K}$ of the chain $\mathbf{m}$ under P. Let $\pi = (\pi_1, \pi_2, \ldots, \pi_K)'$ be the initial distribution of the chain, where $\pi_i = \Pr(\mathbf{m_0} = \mathbf{e_i})$. Assume that the process is stationary. Elliott et al. (1994) proved that the chain $\mathbf{m}$ admits the following decomposition under P:

$$\mathbf{m_{t+1}} = \mathbf{A}\mathbf{m_t} + \mathbf{M_{t+1}}$$

where $\mathbf{M_{t+1}}$ is a martingale with respect to the filtration generated by $\mathbf{m}$ under P.

The financial assets. Assume there are two financial assets available on the economy: a risk-free asset or bond traded at price B_t and a risky asset or stock traded at price S_t at time t. Let r_i be the return of the risk-free asset when the economy is in state i and let $\mathbf{r} = (r_1, \ldots, r_K)$. The return of the risk-free asset at time t depends on the state of the economy at time $t-1$, then $r_t =< \mathbf{r}, \mathbf{m_{t-1}} >$, where $<,>$ denotes the product of vectors. The evolution of the price of the risk-free asset over time is $B_{t+1} = B_t \exp(r_{t+1})$. Assume that the dynamics of the price of the risky asset is given by

$$S_{t+1} = S_t \exp(\mu_t^S - \frac{1}{2}\left(\sigma_t^S\right)^2 + \sigma_t^S \epsilon_{t+1}^S) \tag{1}$$

where μ_t^S and σ_t^S denote, respectively, the return and the volatility of the risky asset and ϵ_{t+1}^S follows a standard normal distribution. The parameters vary across the regimes ($i = 1, \ldots, K$) driven by the chain process $\mathbf{m}$. The vectors containing the return and the volatility of the risky asset for the K states of the economy are, respectively, $\boldsymbol{\mu}^S = (\mu_1^S, \ldots, \mu_K^S)$ and $\boldsymbol{\sigma}^S = (\sigma_1^S, \ldots, \sigma_K^S)$. Then, the parameters of the equation (1) can be rewritten as

$$\mu_t^S =< \boldsymbol{\mu}^S, \mathbf{m_{t-1}} >; \sigma_t^S =< \boldsymbol{\sigma}^S, \mathbf{m_{t-1}} > . \tag{2}$$

Labor income. Assume that individuals live until date T and they receive a labor income amount $L_t > 0$ during their working life period for the years $\{0, 1, 2, \ldots, T^*\}$, where $T^* < T$

denotes the retirement date [2].

Let μ^L be the annual growth rate of the labor income and σ^L be its deviation. Assume that the dynamics of the labor income process are given by

$$L_{t+1} = L_t \exp(\mu_t^L + \sigma_t^L \epsilon_{t+1}^L) \tag{3}$$

where ϵ_{t+1}^L follows a standard normal distribution and the parameters of the equation (3) may switch across the K different states of the economy over time, thus

$$\mu_t^L =< \boldsymbol{\mu}^L, \mathbf{m_{t-1}} >; \sigma_t^L =< \boldsymbol{\sigma}^L, \mathbf{m_{t-1}} > . \tag{4}$$

where $\boldsymbol{\mu}^L = (\mu_1^L, \ldots, \mu_K^L)$ and $\boldsymbol{\sigma}^L = (\sigma_1^L, \ldots, \sigma_K^L)$. The processes given in equations (1) and (3) are correlated, reflecting the fact that the dynamics of the financial market affects the labor income of the individual. Thus,

$$\epsilon_{t+1}^L = \rho \epsilon_{t+1}^S + \sqrt{1-\rho^2}\epsilon_{t+1} \tag{5}$$

where ρ denotes the correlation between the labor income process and the risky asset price process and ϵ_{t+1} follows a standard normal distribution independent of ϵ_{t+1}^S.

Equations (3) and (5) capture the labor income risk profiles across individuals that may share the same personal characteristics but that differ on their labor income risk. If $\sigma^L = 0$ the labor income of the individual behaves as a risk-free bond, while if $\rho = 1$ the labor income behaves as a stock. Section 4 will compare optimal asset-allocation results for different labor risk profiles.

Human capital. Let H_t be the value of human capital at time t, that is, the present value of the future labor income of the individual during his remaining working life period $\{t+1, t+2, \ldots, T^*\}$. H_t is calculated by[3]

$$H_t = \sum_{i=t+1}^{T^*} L_i \exp(-\sum_{j=t+1}^{i} d_j) \tag{6}$$

where d_j denotes the stochastic rate at which labor income is discounted at each period $j = t + 1, t+2, \ldots, T^*$. The discount rate includes the risk-free rate r_j plus the risk premium for the labor income process κ_j, i.e. $d_j = r_j + \kappa_j$. According to the CAPM, κ_j can be evaluated as $\kappa_j = \rho\sigma_j^L/\sigma_j^S(\mu_j^S - r_j)$.

Individual preferences. Let $U(c)$ be the utility function of the individual over his whole life period. The utility function U can be written as

$$U(c) \equiv E\left[\sum_{t=0}^{T} \exp(-\beta t) u(c_t)\right] \tag{7}$$

where $u(c_t)$ is the utility of the individual at each period discounted at a constant subjective rate β. For numerical purposes assume that the individual has a CRRA utility function that takes the form $u(y) = \frac{y^{1-\gamma}}{1-\gamma}$ for $y > 0, \gamma \neq 1$ and $u(y) = \ln(y)$ for $y > 0, \gamma = 1$; where γ denotes the risk aversion parameter.

[2] In a more realistic framework, survival probabilities can be incorporated in the model to allow T to be stochastic. See for example Blake et al. (2003). For simplicity, this paper assumes that T^* and T are known in advance to isolate the effect that labor income risk and change of states in the economy have upon optimal asset-allocation.

[3] For $K = 1$ and $\rho = 1$, i.e. no regime switching and perfect correlation between market and labor income, H_t is the present value of an annuity that increases as a geometric progression with initial payment L_t and growth rate $\mu^L - \sigma^L(\mu^S - r)/\sigma^S$.

3. THE OPTIMAL ASSET-ALLOCATION PROBLEM

During the accumulation period, at each period t, the individual works and receives in exchange a labor income amount L_t and he has to decide: how much to consume $c_t > 0$; the proportion α_t of his remaining wealth that he will allocate to stocks, and the proportion $(1-\alpha_t)$ that he will allocate to bonds, where $0 \leq \alpha_t \leq 1$. The evolution of wealth is given by

$$W_{t+1} = (W_t + L_t - c_t)[\alpha_t \exp(\mu_t^S - \frac{1}{2}\left(\sigma_t^S\right)^2 + \sigma_t^S \epsilon_{t+1}^S) + (1-\alpha_t)\exp(r_t)] \tag{8}$$

The problem is to find optimal α_t and c_t that maximize the individual's utility of total wealth W_t and human capital H_t over his life period. The Bellman equation is written as

$$V(W_t + H_t) = \max\left[U(c_t) + E_t V_{t+1}(W_{t+1} + H_{t+1})\right] \tag{9}$$

where V_t is the value function, W_t is the state variable described in equation (8) and $\{c_t, \alpha_t\}$ are the control variables of the dynamic optimization problem.

Equation (9) has no analytical solution so it will be solved numerically by backwards induction.

4. NUMERICAL IMPLEMENTATION AND RESULTS

The numerical procedure is the following: first, generate the discrete Markov chain $\mathbf{m_t}$ that describes the different states of the economy by using the initial probability distribution of the chain π and the transition probability matrix $\mathbf{A}$. Then simulate the values of the financial assets by using equations (1) and (2). Then compute ϵ_{t+1}^L by using equation (5) and simulate the labor income process L_t by using equation (3). Calculate the human capital H_t by using equation (6). Simulate this procedure N times and evaluate the objective function as an average of equation (7). Solve equation (9) by backwards induction.

The benchmark parameters for the numerical simulations are the following: $N = 10\,000$ simulations; $K = 2$ states of the economy; $T^* = 40$; $T = 65$ years[4]; annual risk-free rate $\mathbf{r} = (0.04, 0.02)$; annual mean of the stock return $\boldsymbol{\mu}^S = (0.10, 0.06)$; standard deviation of the stock return $\boldsymbol{\sigma}^S = (0.12, 0.20)$; annual growth rate of the labor income[5] $\boldsymbol{\mu}^L = (0.01, 0.01)$; standard deviation of the labor income $\boldsymbol{\sigma}^L = (0.03, 0.03)$; correlation between market and labor income $\rho = \{0, 0.2, 0.5, 0.8, 1\}$.

Optimal asset-allocation for individuals that exhibit different labor income risk profiles are analyzed. Although labor income parameters remain the same over the two states of the economy, the regime switching on the market evolution affects the dynamics of the labor income by the correlation parameter. Human capital for different levels of correlation is depicted in Figure 1. It is clear that human capital decreases with the age, but when two states of the economy are considered, human capital may decrease faster or slower according to the prevalent state of the economy.

[4]It corresponds, for example, to an individual that starts working at age 25, retires at age 65 and dies at age 90.

[5]For simplicity, assume that labor income parameters are the same across states. However, different safety *vs.* risky labor profiles are analized according to their labor-market correlation.

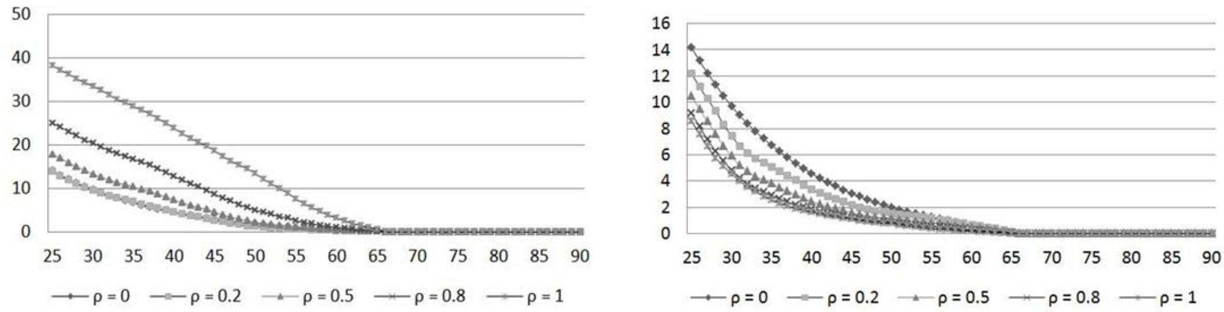

Figure 1: Human capital for different correlation levels: (left) market mostly on expansion; (right) market on recession followed by expansion periods

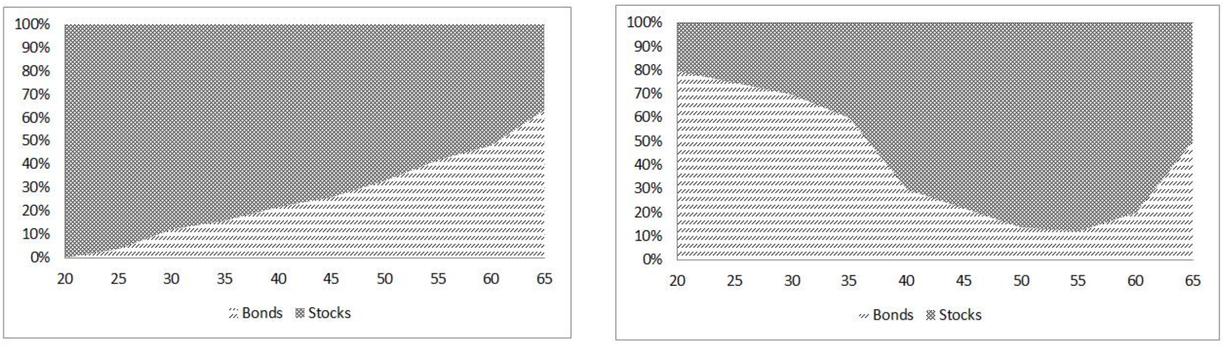

Figure 2: Optimal asset-allocation: (left) safety labor income profile, market mostly on expansion; (right) risky labor income profile, market on recession followed by expansion periods

Figure 2 (left) depicts optimal asset-allocation results when the labor income is not correlated to the market and the market is mostly in an expansion state. In the early years, the individual invests mostly in stocks because his labor income is bond-likely, afterwards he reduces it gradually as his human capital also decreases.

If the economy is in a recession state during the period the individual is young, and if his labor income is highly correlated with the market, a very different optimal portfolio is derived as it is depicted in Figure 2 (right): in the early years, since his labor income is highly correlated with the market and the market is mostly in recession, it is optimal for him to hold mostly bonds. Then, when the economy switches to an expansion state, he is better off by holding mostly stocks, but later, as his human capital decreases with the age, he will hold mostly bonds before his retirement date.

5. CONCLUSIONS

By considering a Markov regime-switching model, this paper analyzes optimal asset-allocation over the life cycle of an individual considering both the evolution of the macroeconomic conditions that affect financial assets and the evolution of the labor income the individual receives.

Numerical simulations show that the traditional advice of holding stocks during the early years and gradually change them to bonds as the individual gets older, is not optimal if the labor income of the individual is imperfectly correlated to the market and the economy exhibits change of regimes. These results should be considered when designing pension plans.

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DYNAMIC HEDGING OF LIFE INSURANCE RESERVES

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1. INTRODUCTION

Longevity risk is the risk associated to the uncertainty about the mortality intensity of the insured sample. The traditional actuarial practice uses deterministic models to describe the evolution of the mortality pattern. More recently, stochastic models for mortality intensity have been introduced, in order to account for unexpected changes in the force of mortality of insured people, given the evidence that, in the last decades, life expectancy of individuals increased more than predicted. Both discrete-time models like the Lee-Carter and its modifications and continuous-time ones (Dahl (2004), Biffis (2005)) have been proposed to price and hedge insurance contracts in the presence of longevity risk. In this paper I provide dynamic hedging strategies for pure endowments in complete and incomplete markets when longevity risk, modelled through a continuous-time and generation-dependent stochastic mortality process, is present.
The topic of hedging in the presence of longevity risk has been addressed in particular in the last ten years, focusing (e.g. Dahl and Møller (2006)) on indifference pricing and risk-minimizing hedging strategies. In a recent paper, Luciano et al. (2012a) derived closed-form expressions for Delta-Gamma neutral hedging strategies under no-arbitrage for the reserves of pure endowment contracts subject to both longevity and financial risk. The paper showed how to assess the risk exposure and to hedge it using either:

1. pure endowments written on the same generation, but with different maturity,
2. mortality-linked products such as longevity bonds also written on the same generation.

In another paper, Luciano et al. (2012b) focus on annuities and term assurances pricing and hedging in the same framework, highlighting the natural hedging opportunities of a life-insurance portfolio within and across cohorts. In this paper we focus again on the hedging of the liability associated with a pure endowment:

1. in a continuous-time dynamic setting;
2. in the presence of longevity risk;
3. accounting for the existence of a source of market incompleteness, namely the absence of hedging instruments whose dynamics is perfectly correlated with the one of the pure endowment we want to hedge. In such case we use pure endowments or longevity bonds written on a different cohort as hedging instruments, i.e, we hedge in the presence of basis risk.

The aim of this paper is to derive closed-form optimal dynamic hedging strategies, which are either perfect hedges when the market is complete or which minimize the variance of the hedging error when basis risk is present. We derive these expressions and we highlight, following Basak and Chabakauri (2012) that the difference between complete and incomplete market hedges is crucially related to the cost of hedging.

2. LONGEVITY RISK MODEL

We consider the following generation-dependent purely diffusive stochastic process for the evolution of the mortality intensity:

$$d\lambda_x(t) = a_x\lambda_x(t)dt + \sigma_x dW_x^M(t) \tag{1}$$

where a_x, $\sigma_x > 0$ and the subscript x highlights that the intensity has a different dynamics depending on a cohort. This is an Ornstein-Uhlenbeck process without mean-reversion, particularly appropriate to model human mortality.[1] It is an affine process which is endowed with the nice analytical properties we highlight in the next section. For pricing purposes, we perform a change of measure selecting a measure - with constant risk premium such that $a'_x = a_x + q$ - which preserves the affine form of the processes involved.[2] Under this measure, no-arbitrage in the market for mortality-linked contracts is guaranteed. Without loss of generality, from now on we consider $q = 0$ and we let our pricing measure coincide with the historical one.

3. DYNAMICS OF THE SURVIVAL PROBABILITY

Since (1) is an affine model, we recall that the survival probability of an individual of cohort x at time t can be written as

$$S_x(t,T) = e^{\alpha(T-t)+\beta(T-t)\lambda_x(t)}$$

where

$$\alpha(T-t) = \frac{\sigma^2}{2a_x^2}(T-t) - \frac{\sigma_x^2}{a_x^3}e^{a_x(T-t)} + \frac{\sigma^2}{4a_x^3}e^{a_x(T-t)} + \frac{3\sigma_x^2}{4a_x^3}$$

$$\beta(T-t) = \frac{1-e^{a_x(T-t)}}{a_x}$$

Notice that we can also provide a convenient closed-form representation of the survival probability at each point in time $t < T$:

$$S_x(t,T) = \frac{S_x(0,T)}{S_x(0,t)}e^{-X(t,T)I(t)-Y(t,T)} \tag{2}$$

where $I(t) = \lambda_x(t) - f_x(0,t)$ is the risk factor against which we hedge. The coefficients have closed-form expressions depending on the parameters of the λ process. The risk factor has an intuitive interpretation, since it is the difference between the actual mortality intensity and its best forecast, which is the so-called forward mortality rate. Applying Ito's lemma, we get the dynamics of the survival probability from time t

[1]See Luciano and Vigna (2008) for an empirical investigation of the appropriateness of non mean reverting affine models for mortality.

[2]See Luciano et al. (2012a) for the details.

to T:

$$\begin{aligned} dS_x(t,T) &= \frac{\partial S}{\partial t}dt + \frac{\partial S}{\partial \lambda}d\lambda_x + \frac{1}{2}\frac{\partial^2 S}{\partial \lambda^2}d\lambda d\lambda \\ &= S_x(t,T)\left(-\alpha'(T-t) - \beta'(T-t)\lambda(t)\right)dt + \beta(T-t)S_x(t,T)a_x\lambda_x dt \\ &\quad +\beta(T-t)S(t,T)\sigma_x dW_x^M + \frac{1}{2}\beta^2(T-t)S(t,T)\sigma_x^2 dt \\ &= S(t,T)\lambda(t)dt + \sigma_x S(t,T)\beta(T-t)dW_x^M(t), \end{aligned}$$

where $\alpha'(\cdot)$, $\beta'(\cdot)$ denote first order derivatives with respect to time and the last equality follows from

$$\begin{cases} \beta(t) = -1 + a_x\beta(t), \\ \alpha(t) = \frac{1}{2}\sigma_x^2\beta^2(t). \end{cases}$$

Summarizing, the dynamics of the survival probability, which coincides with the time-t reserve of the pure endowment, can be written as

$$\frac{dS_x(t,T)}{S_x(t,T)} = \lambda_x(t)dt + \sigma_x\frac{1-e^{a_x(T-t)}}{a_x}dW_x^M. \tag{3}$$

We can derive a similar expression for the reserves of pure endowments written on another cohort i. Assuming correlation between the Brownian motions involved, we can write:

$$\frac{dS_i(t,T)}{S_i(t,T)} = \lambda_i(t)dt + \sigma_i\frac{1-e^{a_i(T-t)}}{a_i}dW_i^M, \tag{4}$$

with $\langle dW_x^M, dW_i^M\rangle = \rho dt$.

4. OPTIMAL DYNAMIC HEDGING STRATEGIES

In Luciano et al. (2012a) we provided closed-form Delta-Gamma hedges for the static problem of covering the reserves of a pure endowment written on a generation of insureds. The hedging strategies described in the paper involve the use of mortality-linked contracts (longevity bonds) and are perfect dynamic hedges provided that the market is complete, in the sense that

1. the hedges are continuously derived and implemented;
2. a sufficient number of hedging instruments written on the same cohort is present.

In this paper we analyze the departure from this second hypothesis. We consider an insurer who has issued a pure endowment written on cohort x and hedges it using as an instrument a pure endowment on another generation. For simplicity, we neglect at this stage the effect of financial risk and consider longevity risk only. The deterministic interest rate is assumed constant and equal to r. We consider then that the reserve $S_x(t,T)$ associated to the pure endowment issued by the insurer is a non-tradable liability, whose associated longevity risk can be partially hedged using pure endowments on another cohort y, which are traded on the market. This is of course a way of dealing with basis risk. If θ_t denotes the dollar amount invested in $S_y(t,T)$, the dynamics of the wealth R_t invested in the money market account and in the pure endowment written on generation y follows the stochastic process:

$$dR_t = \left[rR_t + \theta_t(\lambda_y(t) - r)\right]dt + \theta_t\sigma_y\frac{1-e^{a_y(T-t)}}{a_y}dW_y^M. \tag{5}$$

Our aim is to find the hedging strategy, i.e. the optimal dollar amount θ^* invested in the pure endowment on generation y which minimizes the variance of the hedging error. Formalizing, we solve

$$\min_{\theta_t} \mathrm{Var}_t \left[-\mathbf{1}_{\{\tau_x > T\}} - R_T \right]$$
$$\text{s.t.(5).}$$

Notice that, in a complete market, in which the risk-neutral measure is unique, the optimal policy would be:

$$\theta_t^* = -\rho \frac{\frac{\sigma_x}{a_x}\left(1 - e^{a_x(T-t)}\right)}{\frac{\sigma_y}{a_y}\left(1 - e^{a_y(T-t)}\right)} S_x(t,T) e^{-r(T-t)}. \tag{6}$$

Notice that the negative sign is due to the fact that we are hedging a negative position on $S_x(t,T)$. In such case, i.e. when $|\rho| = 1$, the hedging would be perfect and we would have no hedging error. Moreover, this strategy coincides coincides with a delta-hedging strategy which covers against the common risk factor across cohorts when $|\rho| \neq 1$ (see Luciano et al. (2012b)).
Following Basak and Chabakauri (2012) we can express the optimal hedging policy in a compact form also when the market is incomplete:

$$\theta_t^* = -\left(\rho \frac{\frac{\sigma_x}{a_x}\left(1 - e^{a_x(T-t)}\right)}{\frac{\sigma_y}{a_y}\left(1 - e^{a_y(T-t)}\right)} S_x(t,T) e^{-r(T-t)} \frac{\partial E_t^*\left[\mathbf{1}_{\{\tau_x > T\}}\right]}{\partial S_x(t,T)} + S_y(t,T) \frac{\partial E_t^*\left[\mathbf{1}_{\{\tau_y > T\}}\right]}{\partial S_y(t,T)} \right) \tag{7}$$

where τ_x and τ_y represent the death arrival time for the individuals belonging to generations x and y respectively. The probability measure $\mathbb{P}^*$ involved in the calculation is a "hedge-neutral" measure under which the two dynamics of $S_x(t,T)$ and $S_y(t,T)$ are still driven by correlated Brownian motions, but the dynamic of the hedging instrument has drift coefficient r. The hedge neutral measure and the risk neutral measure we defined above coincide only in the very special case when $\lambda_y(t) - r = 0$. In that case the hedging strategy coincides with (6).
In more general cases, the computation of the Greeks appearing in (7) is not an easy task, since under the hedge neutral measure the λ_x and λ_y processes are not affine anymore. However, Basak and Chabakauri (2012) provide us with the following enlightening link between the two expected values of the payoffs at time t under the different measures:

$$E_t[\mathbf{1}_{\{\tau_x > T\}} e^{-r(T-t)}] - E_t^*[\mathbf{1}_{\{\tau_x > T\}} e^{-r(T-t)}] = E_t[R_T^* e^{-r(T-t)} - R_t],$$

which shows that the difference between the complete market hedge and the incomplete market one is due to the expected difference between discounted terminal and actual wealth, i.e. to the cost of hedging from time t to maturity. It is also possible to compute the hedging error variance:

$$\mathrm{Var}_t[\mathbf{1}_{\{\tau_x > T\}} - R_T^*] = \left(1 - \rho^2\right) E_t \left[\int_t^T \left[\frac{\sigma_x}{a_x} \left(1 - e^{a_x(T-s)}\right) \right]^2 S_x(s,T)^2 \left(\frac{\partial E_s^*[\mathbf{1}_{\{\tau_x > T\}}]}{\partial S_x(s,T)} \right)^2 ds \right].$$

5. CONCLUSIONS

Using a cohort-based stochastic mortality model we derived complete market dynamic hedges of a pure endowment reserve, when the hedging instrument is a pure endowment written on a cohort whose mortality intensity is perfectly correlated with the original one. Following Basak and Chabakauri (2012) we derived also the hedges when the market is incomplete, i.e. when basis risk arise. We highlighted that the difference between the hedges is related to the expected cost of hedging by means of the correlated instrument. Applications of these results to a market in which both longevity and financial risks are present is in the agenda for future research.

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DYNAMIC ASSET ALLOCATION OF BRAZILIAN ACTUARIAL FUNDS

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1. INTRODUCTION

Under Brazilian law, managers of insurance companies and pension funds are allowed to purchase shares of investment funds for the purposes of easing their operations, avoiding liquidity risks and reducing transaction costs. This allowance sped the establishment of several investment funds (commonly known as actuarial), the main purpose of which is to offer protection against their primary risk, a mismatch between the real values of their assets and liabilities. Given that the liabilities of pension funds are usually inflation-indexed in the Brazilian market, that most securities do not offer direct protection against inflation and that actuarial funds promise to maintain the real value of their portfolios, the demand for the services offered by actuarial funds has surged. This naturally calls for the development of tools to understand, measure and control their activities and strategies to achieve their goals.

The main objective of this work is to use dynamic style analysis to uncover the strategies followed by Brazilian actuarial funds from January 2004 to August 2008 and to investigate whether manager's decisions were compatible with the goal of protecting the investor against the negative effects of inflation. The methodology is compatible with time-varying exposures and selectivity skills, which are essential to show how resources have been allocated to the various asset classes available in the Brazilian market during the aforementioned period, and to analyse and judge the performances of fund managers. An important part of the paper discusses how to build and/or choose market indexes capable of characterizing the returns provided by the main securities available because the results depend upon the quality of these indicators. This effort can also be seen as a contribution to the literature, as Pizzinga and Fernandes (2006) assert that such indicators are not easily calculated or readily available. We propose to use a set of indexes throughout this paper that essentially reflect the returns provided by certain representative portfolios, namely those comprising assets that have some common attributes and the returns of which behave similarly.

The models considered in this paper are embedded within a linear state space modelling approach under restrictions, and consequently, their estimations are carried out by way of a restricted Kalman filter under exact initialization combined with (quasi) maximum likelihood estimation. This seems to be the natural solution for the problem at hand, given the time-varying nature of the quantities of interest, i.e., the exposures and the selectivity of the actuarial funds. However, the restriction considered in this paper is that the time-varying exposures must sum-up to 1 (one) for each time instant, commonly termed the portfolio restriction. Imposing this restriction, whenever the data provide evidence that it is appropriate, yields two advantages: (i) a more parsimonious model is considered, and therefore the numerical difficulties of the estimation are minimized and (ii) it tends to create negative correlations between exposures, which is to be expected because fund managers have to decrease the amount invested in some markets (and hence their associated exposures) to increase the amount invested in the preferred markets. Regarding the exact initialization, this method for starting the Kalman filter imparts a greater numerical stability in the estimation (see details in Koopman (1997)).

2. THE METHODOLOGICAL APPROACH

We shall consider the following version of the linear state space model, which does not account for regression and/or intervention effects and is defined by its system matrices $\{Z_t, T_t, H_t, Q_t\}$, which must evolve deterministically:

$$\begin{aligned} Y_t &= Z_t\gamma_t + \epsilon_t \\ \gamma_{t+1} &= T_t\gamma_t + \eta_t \end{aligned} \tag{1}$$

for $t = 1, 2, \ldots, n$. The first equation is know as the measurement equation, and the second is the state equation. The latter describes the dynamics of the state vector γ_t, which is an unobservable m-variate stochastic process and is such that the initial state vector has a finite unconditional mean and covariance matrix, denoted by a_1 and P_1, respectively. If γ_1 is a Gaussian random vector independent of $(\epsilon_t' \, \gamma_t')$ and, additionally,

$$\begin{bmatrix} \epsilon_t \\ \gamma_t \end{bmatrix} \sim NID\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} H_t & 0 \\ 0 & Q_t \end{bmatrix} \right) \tag{2}$$

the model (1) is termed the Gausssian version state space model. For several well-established time series models, there is at least one linear state space representation. A set of recursive formula for obtaining of the state vector, for each time instant and sharing good statistical properties, is the *Kalman filter*. Before discussing such equations, consider a time series of size n of (1) and denote by $\mathcal{F}_j$ the σ-field generated by the measurements until time j: $\mathcal{F}_j \equiv \sigma(Y_1, Y_2, \ldots, Y_j)$. Consider also the second-order conditional moments $\hat{\gamma}_{t|j} \equiv E(\gamma_t|\mathcal{F}_j)$ and $P_{t|j} \equiv E[(\gamma_t - \hat{\gamma}_{t|j})(\gamma_t - \hat{\gamma}_{t|j})'|\mathcal{F}_j]$. The Kalman recursions for the cases of *prediction* $(j = t - 1)$, *updating* $(j = t)$ and *smoothing* $(j = n)$ are:

- *Prediction*

$$\begin{aligned} \hat{\gamma}_{t+1|t} &= T_t \hat{\gamma}_{t|t} \\ P_{t+1|t} &= T_t P_{t|t} T_t' + Q_t \end{aligned} \tag{3}$$

- *Updating*

$$\begin{aligned} \hat{\gamma}_{t|t} &= \hat{\gamma}_{t|t-1} + P_{t|t-1} Z_t' F_t^{-1} \upsilon_t \\ P_{t|t} &= P_{t|t-1} - P_{t|t-1} Z_t' F_t^{-1} P_{t|t-1} \end{aligned} \tag{4}$$

- *Smoothing*

$$\begin{aligned} &\hat{\gamma}_{t|n} = \hat{\gamma}_{t|t-1} + P_{t|t-1} r_{t-1} \\ &r_{t-1} = Z_t' F_t^{-1} \upsilon_t + (T_t - T_t P_{t|t-1} Z_t' F_t^{-1} Z_t)' r_t \\ &P_{t|n} = P_{t|t-1} - P_{t|t-1} N_{t-1} P_{t|t-1} \\ &N_{t-1} = Z_t' F_t^{-1} Z_t + (T_t - T_t P_{t|t-1} Z_t' F_t^{-1} Z_t)' N_t (T_t - T_t P_{t|t-1} Z_t' F_t^{-1} Z_t) \\ &r_n = 0,\ N_n = 0 \ \text{ and } \ t = n, \ldots, 1 \end{aligned} \tag{5}$$

Notice that (3), (4) and (5) require the innovation vectors and their covariance matrices, which are $\upsilon_t \equiv Y_t - E(\gamma_t | \mathcal{F}_{j-1}) = Y_t - Z_t \hat{\gamma}_{t|t-1}$ and $F_t = Z_t P_{t|t-1} Z_t' + H_t$. Finally, even though the Gaussian hypothesis is somewhat restrictive, everything here applies in generality because, for linear non-Gaussian state space models, expressions of the Kalman filter represent optimal linear estimators of the state vector and their corresponding mean square error matrices.

According to (Pizzinga 2010), the reduced restricted Kalman filter is the most natural way of imposing the portfolio restriction on the dynamic asset class factor model (to be discussed later on) properly converted into a linear state space model. Briefly, this method of implementing the Kalman filter under linear restrictions consists of rephrasing some coordinates of the state vector as appropriate affine functions (that is, linear functions plus an intercept term) of the others, placing the result in the measurement equation, and applying the usual Kalman filter with the modified (i.e., reduced) model. Formally, suppose that for some time indices $t \in 1, 2, \ldots, n$, the corresponding state vectors satisfy linear restrictions; that is, $A_t \gamma_t = q_t$, where A_t is a deterministic $k \times m$ matrix and q_t is a (possibly random) observable vector, which is $\mathcal{F}_t$-measurable. The reduced restricted Kalman filter is given in the form of an algorithm:

1. Rewrite the linear restriction as

$$A_{1,t}\gamma_{1,t} + A_{2,t}\gamma_{2,t} = q_t \tag{6}$$

   where $\gamma_{1,t} = (\gamma_{t1}, \ldots, \gamma_{tk})'$ and $A_{1,t}$ has full rank.

2. Solve (6) for $\gamma_{1,t}$ and obtain

$$\gamma_{1,t} = A_{1,t}^{-1} q_t - A_{1,t}^{-1} A_{2,t} \gamma_{2,t} \tag{7}$$

3. Take (7) and place it in the measurement equation in (1)

$$\tilde{Y}_t = Y_t - Z_{1,t} A_{1,t}^{-1} q_t = (Z_{2,t} - Z_{t,1} A_{1,t}^{-1} A_{2,t})\gamma_{2,t} + \epsilon_t = \tilde{Z}_{2,t}\gamma_{2,t} + \epsilon_t \tag{8}$$

4. Postulate a transition matrix equation for the unrestricted state vector $\gamma_{2,t}$ to obtain a reduced linear state space model

$$\gamma_{2,t+1} = T_{2,t}\gamma_{2,t} + \eta_{2,t} \tag{9}$$

5. For the model defined by (8) and (9), apply the usual kalman filter to obtain estimates for $\gamma_{2,t}$ and use them to estimate $\gamma_{1,t}$

$$\hat{\gamma}_{1,t} = A_{1,t}^{-1}q_t - A_{1,t}^{-1}A_{2,t}\hat{\gamma}_{2,t} \quad \forall t \leq j \tag{10}$$

2.1. Dynamic Asset Class Factor Model

Style analyses are implemented through the estimation of *asset class factor models*, the static version (that is, exposures are time-invariant and propose by Sharpe (1988)) of which, equipped with an intercept term (the Jensen's alpha), is

$$R_t^f = \alpha + \beta_1 R_{1,t} + \beta_2 R_{2,t} + \cdots + \beta_k R_{k,t} + \epsilon_t \quad t = 1, 2, \ldots, n \tag{11}$$

where R_t^f is the fund's return, $R_{k,j}$ is the return of the class index j and ϵ_t is a random shock that should be viewed as an idiosyncratic part of the fund's return. For (11), the restrictions generally considered on the exposures $\beta_1 R_{1,t}, \beta_2, \ldots, \beta_k$ are the following: (i)$\sum_{j=1}^{k}\beta_j = 1$ (this is the accounting portfolio restriction) and (ii) $\beta_j \geq 0,\ j = 1, 2, \ldots, k$ (this is the short sale restriction). Sharpe (1992) tackles the estimation of (1) under the enunciated restrictions, minimizing the residual variance, or the sum of squares. Finally, the Jensen's alpha is intended to measure how much the fund manager gains or looses because of his selectivity skills.

A convenient way of incorporating a time-varying structure into a style analysis is to formulate appropriate dynamics for the coefficients featured in model (11). Strictly speaking, one could recognize that the exposures and the Jensen's alpha are unobserved, stochastic processes that may be estimated under a linear state space model using the information obtained from the time series of the returns of the fund and asset class indexes. This represents a dynamic style analysis, and the resulting model could be called a dynamic asset class factor model (see Swinkels and Van Der Sluis (2006) and Pizzinga and Fernandes (2006)). For the weak dynamic style analyses (no restriction is imposed on the exposures) and for the semi-strong dynamic style analyses (the portfolio restriction is assumed) seen in this paper, we assume a random walk evolution for the Jensen's alpha and a stationary vector autoregressive model for the exposures, that is, $\beta_{j,t+1} = \phi_j\beta_{j,t} + \eta_{j,t}$, where $|\phi_j| < 1$ for each j and for $t = 1, 2, \ldots, n$. The portfolio restriction can be implemented by the reduced restricted Kalman filter. The final state space equations derived from the algorithm proposed in (6) to (10) for the dynamic asset class factor model for a semi-strong style analysis are displayed below:

$$\begin{aligned} R_t^f - R_{1,t} &= [R_{2,t} - R_{1,t} \cdots R_{k,t} - R_{1,t}\ 1]\gamma_{2,t} + \epsilon_t \quad \epsilon \sim NID(0, \sigma^2) \\ \gamma_{2,t+1} &= \text{diag}(\phi_2, \phi_3, \ldots, \phi_3, 1) + \eta_{2,t} \quad \eta_{2,t} \sim MVN(0, Q) \end{aligned} \tag{12}$$

where $\gamma_{2,t} \equiv (\beta_{2,t}, \ldots \beta_{k,t}, \alpha_t)'$ and $\gamma_{1,t} \equiv \beta_{1,t} = 1 - [1 \cdots 1\ 0]\gamma_{2,t}$. The covariance matrix Q is considered full, and the Choleski decomposition $Q = CC'$, where C is a triangular matrix with strictly positive diagonal entries c_{ii}, will be used in the empirical examples to guarantee its positive definiteness condition.

3. RESULTS

We presents the results of the estimation of model (19) with the returns of a Brazilian actuarial fund. The asset class indexes should be indicators that reflect the returns of the most basic securities transacted in Brazilian markets (exhaustiveness) without being excessively correlated.

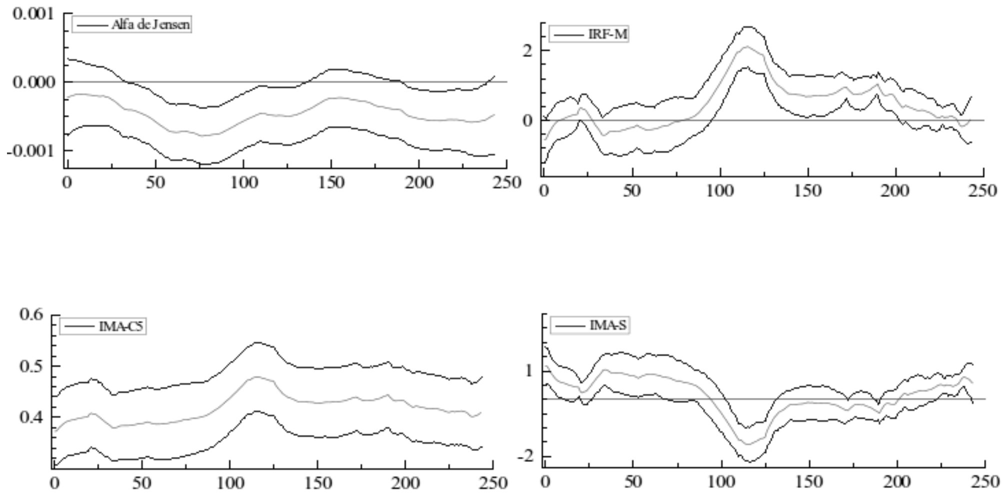

Figure 1: Time plot of the smoothed exposures for CAIXA fund with 95% CI

Figure 1 depicts time plots for the Kalman smoother estimates (with the respective 95% confidence intervals) of Jensen's alpha and the exposures associated with indexes IMA-C5, IRF-M and IMA-S.Visual inspection suggests that its managers assumed a significant long position in IMA-C5 over time, which makes sense because the main objective of any actuarial fund is to offer protection against inflation. Concerning liquidity, however, they operated primarily with bonds of less than 5 years until maturity. The monetary authorities in Brazil eased monetary policy between July 2005 and July 2007, which explains why CAIXA's managers decided to hold a strong long position in IRF-M (this means that they increased the share of conventional bonds in their portfolio) and a strong short position in IMA-S (in other words, they substantially reduced the share of SELIC-indexed bonds) between observations 100 and 150 which pertain mostly to 2006.

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